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# Self Similar Compound Symmetry Covariance Structure

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**Abstract**

Self similar compound symmetry covariance structure is introduced and studied. $k$ SSCS covariance structure (defined in Section 3) for $k$ th order array-variate data incorporates the exchangeable feature of $k$-dimensional arrays into the model. 3 SSCS covariance structure or doubly block compound symmetry (DBCS) covariance structure for array-variate 3rd order data is a generalization of 2 SSCS covariance structure or BCS covariance structure for the matrix-variate 2nd order data, which in turn is a generalization of compound symmetry (CS) covariance structure for traditional vector-variate (multivariate) 1st order data. This article generalizes this CS covariance structure for $k$ th order data, and we name it as "$k$ self similar compound symmetry" (SSCS) covariance structure. This is of critical importance to a variety of applied problems in medical, biomedical, engineering, agricultural and space mechanics among many other fields with $k-$dimensional array-variate data.

**Keywords** Array-variate data; Eigenblock; high dimensional data; Wishart distribution

**JEL Classification** C13; C16

## 1 Introduction

In this paper we introduce self similar compound symmetry (SSCS) covariance structure and study it's properties. SSCS is an interesting family of matrices, but they have not been introduced and studied in previous literature explicitly. SSCS matrix was partially introduced in Leiva (2007) and Roy and Leva (2007) in 2007. This work is originally motivated by modeling the array-variate or matrix-shaped data. A $k-$SSCS matrix is a pattern square matrix which is fully specified by $k$ equal dimensioned unstructured covariance matrices. In this paper, we mainly focus on getting unbiased

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estimates of the eigenblocks of $k-$SSCS matrix and their distributions. Through a series of examples and lemmas we prove many properties of the SSCS matrix.

The singular value decomposition of a matrix is extremely useful for studying matrix-shaped data coming from numerous applications. In light of the excellent properties of the singular value decomposition, and of the occurrence of $k$ th order array-variate data coming from applications, it is a topic of major interest to extend the singular value decomposition to $k-$SSCS matrix. We show through a corollary that the distributions of the eigenblocks for the $k$ th order data with $k$ SSCS covariance structure generalize the distributions of the eigenblocks for the 2nd and 3rd order data with 2 SSCS and 3 SSCS covariance structures respectively.

Statistical methods in many areas of application often require mathematical computations with vectors, matrices and higher order arrays. Hence, matrix theory provides a great tool set for addressing statistical problems. In particular, formulas and algebraic tricks for handling patterned matrices play important roles in the derivations and characterizations of predictors/estimators and performances under general linear models (Arnold, 1979; Roy and Fonseca 2012). A $k-$SSCS covariance structure, is a partitioned covariance matrix, consists of $k$ unstructured covariance matrices for the $k$ arrays of the $k$ th order data, and thus reduces the number of unknown parameters significantly. Eigendecomposition of a suitable structured variance-covariance matrix for the $k$ th order data and to work with the independent principal vectors with the eigenblocks $\big($Hao et al. (2015)$\big)$ as their variance covariance matrices is a way to reduce the burden of dealing with big arrays.

Roy and Leiva (2008), Roy et al. (2015, 2018) have studied 2nd order data with BCS or block exchangeable (2 SSCS) covariance structure. Leiva and Roy (2011, 2012) have also studied 3rd order data with DBCS or doubly exchangeable (3 SSCS) covariance structure. Roy and Fonseca (2012) studied general linear models for 3rd order data with 3 SSCS covariance structure on errors. Recently, Kozioł et al. (2018) studied the optimal properties of 3 SSCS covariance structure for 3rd order data.

The rest of the article is organized as follows. We first set up some preambles about some matrix notations and some definitions related to block matrices and their properties through a lemma and couple of examples in Section 2. We define $k$ SSCS covariance matrix in Section 3. Properties of the SSCS covariance matrix are also discussed in Section 3 through some lemmas and examples. Estimates of the eigenblocks and their distributions are obtained in Section 4, and finally, Section 5 concludes with some remarks.

## 2 Preambles

Let $m_g$, for $g = 1, \ldots, k$, be natural numbers greater than 1, and $p_{i,j}$ be given by

$$p_{i,j} = \begin{cases} \prod_{g=i}^{j} m_g & if \quad 0 \le j - i < k \\ 1 & if \quad j - i = -1 \\ 0 & if \quad j - i = -2 \end{cases}, \tag{1}$$

with $i = 1, \ldots, k$. We denote by $F_g$ the set $F_g = \{1, \ldots, m_g\}$, for $g = 1, \ldots, k$.

**Definition 1.** *We say that a matrix* $\underset{p_{1,k} \times p_{1,k}}{\boldsymbol{A}_k}$ *is a* $k$ *th order block matrix according to the factorization* $p_{1,k} = \prod_{g=1}^{k} m_g$ *to point out that it can be expressed as a* $k$ *different "natural" partitioned matrix forms, that is,*

$$\underset{p_{1,k} \times p_{1,k}}{\boldsymbol{A}_k} = \left( \underset{p_{1,j} \times p_{1,j}}{\boldsymbol{A}_{f_k, \ldots, f_{k+1-(k-j)}; f_k^*, \ldots, f_{k+1-(k-j)}^*}} \right)_{f_k, f_k^* \in F_k; \ldots; f_{j+1}, f_{j+1}^* \in F_{j+1}} : j = 0, \ldots, k-1. \tag{2}$$

Note that for the case $j = 0$ the matrix $\boldsymbol{A}$ is a $(k \times k)-$dimensional matrix with $1 \times 1$ blocks. Clearly, both $m_1 \ge 2$ and $m_2 \ge 2$ for 2nd order data, and $m_1 \ge 2$, $m_2 \ge 2$ and $m_3 \ge 2$ for 3rd order data, and so on. Next we define matrix operators that will be useful in working with these $k$ th order block matrices, where $k \ge 2$. Let $\mathcal{M}_{p_{1,g}}$ denote the set of $p_{1,g} \times p_{1,g}-$ matrices.

**Definition 2.** *Let* $BS_{p_{1,g}}$ *and* $BT_{p_{1,g}}$ *denote respectively the* $p_{1,g} - Sum$ *and* $p_{1,g} - Trace$ *block operators from* $\mathcal{M}_{p_{1,h}}$ *to* $\mathcal{M}_{p_{1,g}}$ *for* $1 \le g \le h \le k$*, where* $\mathcal{M}_{p_{1,h}}$ *will always be evident from the context. These block operators applied to a matrix*

$$\underset{p_{1,g}p_{g+1,h} \times p_{1,g}p_{g+1,h}}{\boldsymbol{G}_k} = \left( \underset{p_{1,g} \times p_{1,g}}{\boldsymbol{G}_{\boldsymbol{f},\boldsymbol{f}^*}} \right)_{\boldsymbol{f},\boldsymbol{f}^* \in F_{h,g+1} = F_h \times \cdots \times F_{g+1} = \times_{j=1}^{h-g} F_{h+1-j}}$$

*give the following* $p_{1,g} \times p_{1,g}-$ *matrices*

$$BS_{p_{1,g}}(\boldsymbol{G}_k) = \sum_{\boldsymbol{f} \in F_{h,g+1}} \sum_{\boldsymbol{f}^* \in F_{h,g+1}} \boldsymbol{G}_{\boldsymbol{f},\boldsymbol{f}^*} \quad and \quad BT_{p_{1,g}}(\boldsymbol{G}_k) = \sum_{\boldsymbol{f} \in F_{h,g+1}} \boldsymbol{G}_{\boldsymbol{f},\boldsymbol{f}}.$$

Note that the subindex $p_{1,g}$ in these block matrix operators represents $p_{1,g} \times p_{1,g}-$dimensional blocks in a partitioned square matrix $\boldsymbol{G}$, and therefore their applications result in $p_{1,g} \times p_{1,g}-$dimensional matrices. In the following lemma we state several interesting and useful properties of these block operators, which we will use later in this article. For any natural number $a > 1$, we use the following additional notations

$$\underset{a \times a}{\boldsymbol{Q}_a} = \boldsymbol{I}_a - \boldsymbol{P}_a, \tag{3}$$

$$\text{and} \quad \boldsymbol{P}_a = \frac{1}{a}\boldsymbol{J}_a, \tag{4}$$

where $\boldsymbol{J}_a = \mathbf{1}_a \mathbf{1}'_a$ with $\mathbf{1}_a$ be the $a \times 1$ vector of ones, and $\boldsymbol{I}_a = [\boldsymbol{e}_{a,1}, \cdots, \boldsymbol{e}_{a,a}]$ be the $a \times a-$ identity matrix with $\boldsymbol{e}_{a,i}$ the $i^{th}$ column vector of $\boldsymbol{I}_a$. Note that $\boldsymbol{P}_a$ and $\boldsymbol{Q}_a$ are idempotent matrices and mutually orthogonal to each other, that is,

$$(\boldsymbol{P}_a)^2 = \boldsymbol{P}_a, \quad (\boldsymbol{Q}_a)^2 = \boldsymbol{Q}_a \quad \text{and} \quad \boldsymbol{P}_a \boldsymbol{Q}_a = \mathbf{0}.$$

**Lemma 1.** *Let* $\underset{p_{1,k} \times p_{1,k}}{\boldsymbol{A}_k}$ *be a symmetric k th order block matrix that can be expressed in its natural form of partitioned matrices as given in (2), and let* $g, h$ *be the natural numbers such that* $1 \leq g \leq h \leq i < k$, *then the following properties hold:*

1. $BT_{p_{1,g}} \left[ BT_{p_{1,h}} \left( \boldsymbol{A}_k \right) \right] = BT_{p_{1,g}} \left( \boldsymbol{A}_k \right).$
2. $BS_{p_{1,g}} \left[ BS_{p_{1,h}} \left( \boldsymbol{A}_k \right) \right] = BS_{p_{1,g}} \left( \boldsymbol{A}_k \right).$
3. $BT_{p_{1,g}} \left[ \sum_{j=1}^{n} \boldsymbol{A}_{j:k} \right] = \sum_{j=1}^{n} BT_{p_{1,g}} \left[ \boldsymbol{A}_{j:k} \right]$ *and* $BS_{p_{1,g}} \left[ \sum_{j=1}^{n} \boldsymbol{A}_{j:k} \right] = \sum_{j=1}^{n} BS_{p_{1,g}} \left[ \boldsymbol{A}_{j:k} \right]$, *where* $\underset{p_{1,k} \times p_{1,k}}{\boldsymbol{A}_{j:k}}$ *is a kth order block matrix for each* $j = 1, \ldots, n$.
4. $BT_{p_{1,g}} \left[ BS_{p_{1,h}} \left( \boldsymbol{A}_k \right) \right] = \sum_{f_k, \ldots, f_{h+1}} \sum_{f_k^*, \ldots, f_{h+1}^*} BT_{p_{1,g}} \left[ \boldsymbol{A}_{k: f_k, \ldots, f_{h+1}; f_k^*, \ldots, f_{h+1}^*} \right].$
5. *For a fixed natural number* $k \geq 2$, *let* $\boldsymbol{R}_{k,j}$ *be the* $p_{1,k} \times p_{1,k}-$*matrix*

$$\boldsymbol{R}_{k,j+1} = \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{I}_{m_1}, \tag{5}$$

*where, for each* $j = 1, \ldots, k-1$,

$$\begin{aligned} \underset{p_{2,k} \times p_{2,k}}{\boldsymbol{R}^*_{k,j+1}} &= \left( \bigotimes_{h=1}^{k-(j+1)} \boldsymbol{I}_{m_{k+1-h}} \right) \otimes \boldsymbol{Q}_{m_{j+1}} \otimes \left( \bigotimes_{h=k-(j-1)}^{k-1} \boldsymbol{P}_{m_{k+1-h}} \right) \\ &= \boldsymbol{I}_{p_{j+2,k}} \otimes \boldsymbol{Q}_{m_{j+1}} \otimes \boldsymbol{P}_{m_j, m_2}, \end{aligned} \tag{6}$$

*with*

$$\boldsymbol{P}_{m_i, m_{i^*}} = \begin{cases} \bigotimes_{h=k-(i-1)}^{k-(i^*-1)} \boldsymbol{P}_{m_{k+1-h}} = \boldsymbol{P}_{m_i} \otimes \boldsymbol{P}_{m_{i-1}} \otimes \cdots \otimes \boldsymbol{P}_{m_{i^*}} & if \quad i \geq i^* \\ 1 & if \quad i < i^* \end{cases} \tag{7}$$

*and* $\bigotimes_{h=1}^{0} \boldsymbol{I}_{m_{k+1-h}} = 1 = \bigotimes_{h=k}^{k-1} \boldsymbol{P}_{m_{k+1-h}}$. *Also, let* $\boldsymbol{R}_{k,k+1}$ *be the* $p_{1,k} \times p_{1,k}-$*matrix*

$$\boldsymbol{R}_{k,k+1} = \boldsymbol{R}^*_{k,k+1} \otimes \boldsymbol{I}_{m_1}, \tag{8}$$

*where*

$$\underset{p_{2,k} \times p_{2,k}}{\boldsymbol{R}^*_{k,k+1}} = \bigotimes_{h=1}^{k-1} \boldsymbol{P}_{m_{k+1-h}} = \boldsymbol{P}_{m_k, m_2}. \tag{9}$$

*Given any* $p_{1,k} \times p_{1,k}-$*matrix* $\boldsymbol{A}_k$ *partitioned into* $p_{1,k-1} \times p_{1,k-1}-$*blocks* $\boldsymbol{A}_{f_k;f_k}$, *that is,*

$$\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_k} = \left(\boldsymbol{A}_{f_k;f_k^*}\right)_{f_k;f_k^*\in F_k},$$

*the following equalities hold:*

$$\begin{aligned} BT_{p_{1,1}}\left[\boldsymbol{R}_{k,j+1}\boldsymbol{A}_k\right] &= \frac{1}{p_{2,j}}\left\{m_j BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{A}_k\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{A}_k\right)\right]\right\} \quad (10)\\ &= BT_{p_{1,1}}\left[\boldsymbol{R}_{k,j+1}\boldsymbol{A}_k\boldsymbol{R}_{k,j+1}\right], \quad \textit{for each} \;\; j=1,\ldots,k-1, \quad \textit{and} \\ BT_{p_{1,1}}\left[\boldsymbol{R}_{k,k+1}\boldsymbol{A}_k\right] &= \frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\boldsymbol{A}_k\right) \quad (11)\\ &= BT_{p_{1,1}}\left[\boldsymbol{R}_{k,k+1}\boldsymbol{A}_k\boldsymbol{R}_{k,k+1}\right]. \end{aligned}$$

*Proof.* Proofs of the Properties 1-4 are straightforward, so we will only prove Property 5 here. We use mathematical induction to prove that equalities (10) and (11) for any natural number $k \geq 2$, i.e., we first prove that these equalities are valid for $k = 2$, and then assuming that they are true for $k$ (Inductive Hypothesis (I.H.)), we prove that they are also true for $k+1$ (Inductive Thesis (I.T.)). Case $k = 2$ : In this case $\boldsymbol{R}^*_{2,j+1}$, for $j = 1, 2$, are, $\boldsymbol{R}^*_{2,2} = \boldsymbol{Q}_{m_2}$ and $\boldsymbol{R}^*_{2,3} = \boldsymbol{P}_{m_2}$ respectively.

(a) We first prove that (11) holds for $k = 2$, that is,

$$\begin{aligned} BT_{p_{1,1}}\left[\left(\boldsymbol{R}^*_{2,3}\otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] &= BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_2}\otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] = \frac{1}{m_2} BS_{m_1}\left(\boldsymbol{A}_2\right) \\ &= BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_2}\otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes \boldsymbol{I}_{m_1}\right)\right]. \end{aligned}$$

Note that

$$\begin{aligned} \left(\boldsymbol{P}_{m_2}\otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2 &= \frac{1}{m_2}\left(\boldsymbol{J}_{m_2}\otimes \boldsymbol{I}_{m_1}\right)\left(\underset{p_{1,1}\times p_{1,1}}{\boldsymbol{A}_{f_2;f_2^*}}\right)_{f_2,f_2^*\in F_2} \\ &= \frac{1}{m_2}\begin{pmatrix} \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;1} & \cdots & \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;m_2} \\ \vdots & \ddots & \vdots \\ \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;1} & \cdots & \sum_{f_2\in F_2}\boldsymbol{A}_{m_2;m_2} \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_2}\otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] &= \frac{1}{m_2}\sum_{f_2\in F_2}\sum_{f_2^*\in F_2}\boldsymbol{A}_{f_2;f_2^*} \\ &= \frac{1}{m_2} BS_{m_1}\left(\boldsymbol{A}_2\right). \end{aligned}$$

Similarly

$$
\begin{aligned}
\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right) &= \frac{1}{m_2^2}\begin{pmatrix} \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;1} & \cdots & \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;m_2} \\ \vdots & \ddots & \vdots \\ \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;1} & \cdots & \sum_{f_2\in F_2}\boldsymbol{A}_{f_2;m_2} \end{pmatrix} \\
&\quad \begin{pmatrix} \boldsymbol{I}_{m_1} & \cdots & \boldsymbol{I}_{m_1} \\ \vdots & \ddots & \vdots \\ \boldsymbol{I}_{m_1} & \cdots & \boldsymbol{I}_{m_1} \end{pmatrix} \\
&= \frac{1}{m_2^2}\boldsymbol{J}_{m_2}\otimes\sum_{f_2\in F_2}\sum_{f_2^*\in F_2}\boldsymbol{A}_{f_2;f_2^*}.
\end{aligned}
$$

Therefore,
$$
\begin{aligned}
BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\right] &= \frac{1}{m_2^2}m_2\sum_{f_2\in F_2}\sum_{f_2^*\in F_2}\boldsymbol{A}_{f_2;f_2^*} \\
&= \frac{1}{m_2}BS_{m_1}\left(\boldsymbol{A}_2\right).
\end{aligned}
$$

(b) To prove (10) holds for $k=2$, that is,

$$
\begin{aligned}
BT_{p_{1,1}}\left[\left(\boldsymbol{R}_{2,2}^*\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] &= BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] \\
&= BT_{p_{1,1}}\left(\boldsymbol{A}_2\right)-\frac{1}{m_2}BS_{m_1}\left(\boldsymbol{A}_2\right) \\
&= \frac{1}{m_2}BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\right],
\end{aligned}
$$

first note that

$$
\begin{aligned}
\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2 &= \left(\boldsymbol{I}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2-\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2 \\
&= \boldsymbol{A}_2-\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2,
\end{aligned}
$$

and using the first equality of Part (a) and the Property 3, we have

$$
\begin{aligned}
BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] &= BT_{p_{1,1}}\left[\boldsymbol{A}_2\right]-BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right] \\
&= BT_{p_{1,1}}\left[\boldsymbol{A}_2\right]-\frac{1}{m_2}BS_{m_1}\left(\boldsymbol{A}_2\right).
\end{aligned}
$$

Now, to prove the second equality note that

$$
\begin{aligned}
\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right) &= \left[\left(\boldsymbol{I}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2-\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\right]\left[\left(\boldsymbol{I}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)-\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\right] \\
&= \boldsymbol{A}_2-2\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)+\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right),
\end{aligned}
$$

then, using part (a) and Property 3 we have

$$
\begin{aligned}
& \left[\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{Q}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\right] \\
=\; & BT_{p_{1,1}}\left[\boldsymbol{A}_2\right]-2BT_{p_{1,1}}\left[\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\right]+BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{A}_2\left(\boldsymbol{P}_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\right] \\
=\; & BT_{p_{1,1}}\left[\boldsymbol{A}_2\right]-\frac{1}{m_2}BS_{m_1}\left(\boldsymbol{A}_2\right).
\end{aligned}
$$

Case $k \Longrightarrow$ *Case* $k+1$ : Assuming (10) and (11) are true, we want to prove the following:

$$\begin{aligned} BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,j+1}\boldsymbol{A}_{k+1}\right] &= \frac{1}{p_{2,j}}\left\{m_j BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\} \\ &= BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,j+1}\boldsymbol{A}_{k+1}\boldsymbol{R}_{k+1,j+1}\right], \quad \text{for each } \ j=1,\ldots,k-1,k \ \text{ and} \\ BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+2}\boldsymbol{A}_{k+1}\right] &= \frac{1}{p_{2,k+1}} BS_{p_{1,1}}\left(\boldsymbol{A}_{k+1}\right) \\ &= BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+2}\boldsymbol{A}_{k+1}\boldsymbol{R}_{k+1,k+2}\right]. \end{aligned}$$

Now for $j=1,\ldots,k-1$, $\boldsymbol{R}_{k+1,j+1} = \boldsymbol{I}_{m_{k+1}} \otimes \boldsymbol{R}_{k,j+1}$ and for $j=k$, $\boldsymbol{R}_{k+1,k+1} = \boldsymbol{Q}_{m_{k+1}} \otimes \boldsymbol{R}_{k,k+1}$. So, for $j=1,\ldots,k-1$, we first have

$$\begin{aligned} & BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,j+1}\boldsymbol{A}_{k+1}\right] \\ = \ & BT_{p_{1,1}}\left[\begin{pmatrix} \boldsymbol{R}_{k,j+1} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{R}_{k,j+1} \end{pmatrix}\begin{pmatrix} \boldsymbol{A}_{1;1} & \cdots & \boldsymbol{A}_{1;m_{k+1}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{A}_{m_{k+1};1} & \cdots & \boldsymbol{A}_{m_{k+1};m_{k+1}} \end{pmatrix}\right] \\ & \overset{\textit{using 1}}{=} BT_{p_{1,1}}\left\{BT_{p_{1,k}}\left[\begin{pmatrix} \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{1;1} & \cdots & \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{1;m_{k+1}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{m_{k+1};1} & \cdots & \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{m_{k+1};m_{k+1}} \end{pmatrix}\right]\right\} \\ = \ & BT_{p_{1,1}}\left\{\sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,j+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}}}\right\} \overset{\textit{using 3}}{=} \sum_{f_{k+1}\in F_{k+1}} BT_{p_{1,1}}\left[\boldsymbol{R}_{k,j+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}}}\right] \\ & \overset{\textit{using I.H.}}{=} \sum_{f_{k+1}\in F_{k+1}} \frac{1}{p_{2,j}}\left\{m_j BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{A}_{f_{k+1};f_{k+1}}\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{A}_{f_{k+1};f_{k+1}}\right)\right]\right\} \\ & \overset{\textit{using 3 and 1}}{=} \frac{1}{p_{2,j}}\left\{m_j BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\}. \end{aligned}$$

The second equality of this case is proved similarly.

$$\begin{aligned} & BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,j}\boldsymbol{A}_{k+1}\boldsymbol{R}_{k+1,j}\right] \\ = \ & BT_{p_{1,1}}\left[\begin{pmatrix} \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{1;1} & \cdots & \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{1;m_{k+1}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{m_{k+1};1} & \cdots & \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{m_{k+1};m_{k+1}} \end{pmatrix}\begin{pmatrix} \boldsymbol{R}_{k,j+1} & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{R}_{k,j+1} \end{pmatrix}\right] \\ & \overset{\textit{using 1}}{=} BT_{p_{1,1}}\left\{BT_{p_{1,k}}\left[\begin{pmatrix} \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{1;1}R_{k,j+1} & \cdots & \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{1;m_{k+1}}\boldsymbol{R}_{k,j+1} \\ \vdots & \ddots & \vdots \\ \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{m_{k+1};1}\boldsymbol{R}_{k,j+1} & \cdots & \boldsymbol{R}_{k,j+1}\boldsymbol{A}_{m_{k+1};m_{k+1}}\boldsymbol{R}_{k,j+1} \end{pmatrix}\right]\right\} \\ = \ & BT_{p_{1,1}}\left\{\sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,j+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}}}\boldsymbol{R}_{k,j+1}\right\} \overset{\textit{using 3}}{=} \sum_{f_{k+1}\in F_{k+1}} BT_{p_{1,1}}\left[\boldsymbol{R}_{k,j+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}}}\boldsymbol{R}_{k,j+1}\right] \\ & \overset{\textit{using I.H.}}{=} \sum_{f_{k+1}\in F_{k+1}} \frac{1}{p_{2,j}}\left\{m_j BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{A}_{f_{k+1};f_{k+1}}\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{A}_{f_{k+1};f_{k+1}}\right)\right]\right\} \\ & \overset{\textit{using 3 and 1}}{=} \frac{1}{p_{2,j}}\left\{m_j BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\}. \end{aligned}$$

In second place, for $j = k$, we have

$$
\begin{aligned}
& BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+1}\boldsymbol{A}_{k+1}\right] \\
= \;& BT_{p_{1,1}}\left\{\left[\boldsymbol{I}_{m_{k+1}} \otimes R_{k,k+1}\right]\boldsymbol{A}_{k+1}\right\} - \frac{1}{m_{k+1}} BT_{p_{1,1}}\left\{\left[\boldsymbol{J}_{m_{k+1}} \otimes \boldsymbol{R}_{k,k+1}\right]\boldsymbol{A}_{k+1}\right\} \\
& \overset{\text{using 1}}{=} BT_{p_{1,1}}\Big\{ \sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}}}\Big\} \\
& -\frac{1}{m_{k+1}} BT_{p_{1,1}}\left\{ BT_{p_{1,k}}\left[\boldsymbol{1}_{m_{k+1}} \otimes \left( \sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};1}}, \dots, \sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};m_{k+1}}} \right)\right]\right\} \\
& \overset{\text{using 3}}{=} \sum_{f_{k+1}\in F_{k+1}} BT_{p_{1,1}}\left( \boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}}} \right) - \frac{1}{m_{k+1}} \sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} BT_{p_{1,1}}\left( \boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f^*_{k+1}}} \right) \\
& \overset{\text{using I.H.}}{=} \sum_{f_{k+1}\in F_{k+1}} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left( \boldsymbol{A}_{f_{k+1};f_{k+1}} \right) - \frac{1}{m_{k+1}} \sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left( \boldsymbol{A}_{f_{k+1};f^*_{k+1}} \right) \\
& \overset{\text{using 3}}{=} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left( \sum_{f_{k+1}\in F_{k+1}} \boldsymbol{A}_{f_{k+1};f_{k+1}} \right) - \frac{1}{m_{k+1}}\frac{1}{p_{2,k}} BS_{p_{1,1}}\left( \sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} \boldsymbol{A}_{f_{k+1};f^*_{k+1}} \right) \\
= \;& \frac{1}{p_{2,k+1}}\left\{ m_{k+1} BS_{p_{1,1}}\left[ BT_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[ BS_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\} \\
& \overset{\text{using 2}}{=} \frac{1}{p_{2,k+1}}\left\{ m_{k+1} BS_{p_{1,1}}\left[ BT_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[\boldsymbol{A}_{k+1}\right]\right\} \\
= \;& \frac{1}{p_{2,k+1}}\left\{ m_{k+1} BS_{p_{1,1}}\left[ BT_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[ BT_{p_{1,k+1}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\},
\end{aligned}
$$

because $BT_{p_{1,k+1}}\left(\boldsymbol{A}_{k+1}\right) = \boldsymbol{A}_{k+1}$.

The second equality is

$$
\begin{aligned}
& BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+1}\boldsymbol{A}_{k+1}\boldsymbol{R}_{k+1,k+1}\right] \\
= \; & BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right)\boldsymbol{A}_{k+1}\left(\boldsymbol{Q}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right)\right] \\
= \; & BT_{p_{1,1}}\left\{\left[\boldsymbol{I}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right]\boldsymbol{A}_{k+1}\left[\boldsymbol{I}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right]\right\} \\
& -\frac{2}{m_{k+1}}BT_{p_{1,1}}\left\{\left[\boldsymbol{J}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right]\boldsymbol{A}_{k+1}\left[\boldsymbol{I}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right]\right\} \\
& +\frac{1}{m_{k+1}^2}BT_{p_{1,1}}\left\{\left[\boldsymbol{J}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right]\boldsymbol{A}_{k+1}\left[\boldsymbol{J}_{m_{k+1}}\otimes\boldsymbol{R}_{k,k+1}\right]\right\} \\
= \; & BT_{p_{1,1}}\left\{\left(\boldsymbol{R}_{k,k+1}\boldsymbol{A}_{f_{k+1};f_{k+1}^*}\boldsymbol{R}_{k,k+1}\right)_{f_{k+1},f_{k+1}^*\in F_{k+1}}\right\} \\
& -\frac{2}{m_{k+1}}BT_{p_{1,1}}\left[\boldsymbol{1}_{m_{k+1}}\otimes\right. \\
& \left.\left(\sum_{f_{k+1}\in F_{k+1}}\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};1}}R_{k,k+1},\ldots,\sum_{f_{k+1}\in F_{k+1}}\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};m_{k+1}}}\boldsymbol{R}_{k,k+1}\right)\right] \\
& +\frac{1}{m_{k+1}^2}BT_{p_{1,1}}\left[\boldsymbol{J}_{m_{k+1}}\otimes\sum_{f_{k+1}\in F_{k+1}}\sum_{f_{k+1}^*\in F_{k+1}}\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}^*}}\boldsymbol{R}_{k,k+1}\right] \\
\overset{\text{using 1}}{=} \; & BT_{p_{1,1}}\left\{BT_{p_{1,k}}\left[\left(\boldsymbol{R}_{k,k+1}\boldsymbol{A}_{f_{k+1};f_{k+1}^*}\boldsymbol{R}_{k,k+1}\right)_{f_{k+1},f_{k+1}^*\in F_{k+1}}\right]\right\} \\
& -\frac{2}{m_{k+1}}BT_{p_{1,1}}\left\{BT_{p_{1,k}}\left[\boldsymbol{1}_{m_{k+1}}\otimes\right.\right. \\
& \left.\left.\left(\sum_{f_{k+1}\in F_{k+1}}\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};1}}\boldsymbol{R}_{k,k+1},\ldots,\sum_{f_{k+1}\in F_{k+1}}\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};m_{k+1}}}\boldsymbol{R}_{k,k+1}\right)\right]\right\} \\
& +\frac{1}{m_{k+1}^2}BT_{p_{1,1}}\left\{BT_{p_{1,k}}\left[\boldsymbol{J}_{m_{k+1}}\otimes\sum_{f_{k+1}\in F_{k+1}}\sum_{f_{k+1}^*\in F_{k+1}}\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}^*}}\boldsymbol{R}_{k,k+1}\right]\right\} \\
\overset{\text{using 3}}{=} \; & \sum_{f_{k+1}\in F_{k+1}}BT_{p_{1,1}}\left(R_{k,k+1}\boldsymbol{A}_{f_{k+1};f_{k+1}}\boldsymbol{R}_{k,k+1}\right) \\
& -\frac{2}{m_{k+1}}\sum_{f_{k+1}\in F_{k+1}}\sum_{f_{k+1}^*\in F_{k+1}}BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}^*}}\boldsymbol{R}_{k,k+1}\right) \\
& +\frac{m_{k+1}}{m_{k+1}^2}\sum_{f_{k+1}\in F_{k+1}}\sum_{f_{k+1}^*\in F_{k+1}}BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1}\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f_{k+1}^*}}\boldsymbol{R}_{k,k+1}\right).
\end{aligned}
$$

Now, using the Inductive Hypothesis (I.H.), we obtain

$$\begin{aligned}
& BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+1}\boldsymbol{A}_{k+1}\boldsymbol{R}_{k+1,k+1}\right] \\
& \overset{\text{using I.H.}}{=} \sum_{f_{k+1}\in F_{k+1}} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\boldsymbol{A}_{f_{k+1};f_{k+1}}\right) \\
& -\frac{1}{m_{k+1}} \sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\boldsymbol{A}_{f_{k+1};f^*_{k+1}}\right) \\
& \overset{\text{using 3}}{=} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\sum_{f_{k+1}\in F_{k+1}} \boldsymbol{A}_{f_{k+1};f_{k+1}}\right) \\
& -\frac{1}{m_{k+1}}\frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} \boldsymbol{A}_{f_{k+1};f^*_{k+1}}\right) \\
= \; & \frac{1}{p_{2,k+1}}\left\{m_{k+1} BS_{p_{1,1}}\left[BT_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[BS_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\} \\
& \overset{\text{using 2}}{=} \frac{1}{p_{2,k+1}}\left\{m_{k+1} BS_{p_{1,1}}\left[BT_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[\boldsymbol{A}_{k+1}\right]\right\} \\
= \; & \frac{1}{p_{2,k+1}}\left\{m_{k+1} BS_{p_{1,1}}\left[BT_{p_{1,k}}\left(\boldsymbol{A}_{k+1}\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,k+1}}\left(\boldsymbol{A}_{k+1}\right)\right]\right\},
\end{aligned}$$

where again we use the equality $BT_{p_{1,k+1}}\left(\boldsymbol{A}_{k+1}\right) = \boldsymbol{A}_{k+1}$. Finally, it only rest to prove the implication

$$BT_{p_{1,1}}\left[\boldsymbol{R}_{k,k+1}\boldsymbol{A}_k\right] = \frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\boldsymbol{A}_k\right) = BT_{p_{1,1}}\left[\boldsymbol{R}_{k,k+1}\boldsymbol{A}_k\boldsymbol{R}_{k,k+1}\right]$$

$$\Longrightarrow$$

$$BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+2}\boldsymbol{A}_{k+1}\right] = \frac{1}{p_{2,k+1}} BS_{p_{1,1}}\left(\boldsymbol{A}_{k+1}\right) = BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+2}\boldsymbol{A}_{k+1}\boldsymbol{R}_{k+1,k+2}\right].$$

Noting that $\quad \boldsymbol{R}_{k+1,k+2} = \boldsymbol{P}_{m_{k+1}} \otimes \boldsymbol{R}_{k,k+1} = \frac{1}{m_{k+1}} \boldsymbol{J}_{m_{k+1}} \otimes \boldsymbol{R}_{k,k+1},$

we have

$$\begin{aligned}
BT_{p_{1,1}}\left[\boldsymbol{R}_{k+1,k+2}\boldsymbol{A}_{k+1}\right] = \; & \frac{1}{m_{k+1}} BT_{p_{1,1}}\left[\left(\boldsymbol{J}_{m_{k+1}} \otimes \boldsymbol{R}_{k,k+1}\right)\boldsymbol{A}_{k+1}\right] \\
& \overset{\text{using 1}}{=} \frac{1}{m_{k+1}} BT_{p_{1,1}}\left\{BT_{p_{1,k}}\left[\boldsymbol{1}_{m_{k+1}} \otimes \right.\right. \\
& \left.\left.\left(\sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,k+1} \underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};1}}, \ldots, \sum_{f_{k+1}\in F_{k+1}} \boldsymbol{R}_{k,k+1} \underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};m_{k+1}}}\right)\right]\right\} \\
& \overset{\text{using 3}}{=} \frac{1}{m_{k+1}} \sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1} \underset{p_{1,k}\times p_{1,k}}{\boldsymbol{A}_{f_{k+1};f^*_{k+1}}}\right) \\
& \overset{\text{using I.H.}}{=} \frac{1}{m_{k+1}} \sum_{f_{k+1}\in F_{k+1}} \sum_{f^*_{k+1}\in F_{k+1}} \frac{1}{p_{2,k}} BS_{p_{1,1}}\left(\boldsymbol{A}_{f_{k+1};f^*_{k+1}}\right) \\
& \overset{\text{using 3 and 2}}{=} \frac{1}{p_{2,k+1}} BS_{p_{1,1}}\left(\boldsymbol{A}_{k+1}\right).
\end{aligned}$$

The second equality of this case can be proved similarly. □

The results of Lemma 1 are particularized for two interesting cases in the following examples:

**Example 1** (Case $k = 3$):

1. $BT_{p_{1,1}}\left[\left(\boldsymbol{I}_{m_3} \otimes \boldsymbol{Q}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_3\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{I}_{m_3} \otimes \boldsymbol{Q}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_3\left(\boldsymbol{I}_{m_3} \otimes \boldsymbol{Q}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right]$

   $= BT_{m_1}\left(\boldsymbol{A}_3\right) - \frac{1}{m_2} BS_{m_1}\left[BT_{p_{1,2}}\left(\boldsymbol{A}_3\right)\right].$

2. $BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_3\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_3\left(\boldsymbol{Q}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right]$

   $= \frac{1}{m_2} BS_{m_1}\left[BT_{p_{1,2}}\left(\boldsymbol{A}_3\right)\right] - \frac{1}{m_2 m_3} BS_{m_1}\left(\boldsymbol{A}_3\right).$

3. $BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_3\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_3\left(\boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right] = \frac{1}{m_2 m_3} BS_{m_1}\left(\boldsymbol{A}_3\right).$

**Example 2** (Case $k = 4$):

1. $BT_{p_{1,1}}\left[\left(\boldsymbol{I}_{m_4} \otimes \boldsymbol{I}_{m_3} \otimes \boldsymbol{Q}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{I}_{m_4} \otimes \boldsymbol{I}_{m_3} \otimes \boldsymbol{Q}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\left(\boldsymbol{I}_{m_4} \otimes \boldsymbol{I}_{m_3} \otimes \boldsymbol{Q}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right]$

   $= BT_{m_1}\left(\boldsymbol{A}_4\right) - \frac{1}{m_2} BS_{m_1}\left[BT_{p_{1,2}}\left(\boldsymbol{A}_4\right)\right].$

2. $BT_{p_{1,1}}\left[\left(\boldsymbol{I}_{m_4} \otimes \boldsymbol{Q}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{I}_{m_4} \otimes \boldsymbol{Q}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\left(\boldsymbol{I}_{m_4} \otimes \boldsymbol{Q}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right]$

   $= \frac{1}{m_2} BS_{m_1}\left\{BT_{p_{1,2}}\left(\boldsymbol{A}_4\right)\right\} - \frac{1}{m_2 m_3} BS_{m_1}\left[BT_{p_{1,3}}\left(\boldsymbol{A}_4\right)\right].$

3. $BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_4} \otimes \boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{Q}_{m_4} \otimes \boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\left(\boldsymbol{Q}_{m_4} \otimes \boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right]$

   $= \frac{1}{m_2 m_3 m_4}\left[m_4 BS_{m_1}\left[BT_{p_{1,3}}\left(\boldsymbol{A}_4\right)\right] - BS_{m_1}\left(\boldsymbol{A}_4\right)\right].$

4. $BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_4} \otimes \boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\right]$

   $= BT_{p_{1,1}}\left[\left(\boldsymbol{P}_{m_4} \otimes \boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\boldsymbol{A}_4\left(\boldsymbol{P}_{m_4} \otimes \boldsymbol{P}_{m_3} \otimes \boldsymbol{P}_{m_2} \otimes \boldsymbol{I}_{m_1}\right)\right]$
   $= \frac{1}{m_2 m_3 m_4} BS_{m_1}\left(\boldsymbol{A}_4\right).$

## 3 Properties of the self similar compound symmetry covariance matrix

Let $\boldsymbol{x}_{r;\boldsymbol{f}}$ be a $m_1-$variate vector of measurements on the $r^{th}$ replicate (individual) at the $\boldsymbol{f} = (f_k, \ldots, f_2) \in F = F_{k,2} = F_k \times \cdots \times F_2 = \underset{g=k}{\overset{2}{\times}} F_g$ factor combination. Let $\boldsymbol{x}_r$ be the $p_{1,k} = \prod_{j=1}^{k} m_j-$variate vector of all measurements corresponding to the $r^{th}$ sample unit of the population, that is, $\boldsymbol{x}_r = (x_{r;1,\ldots,1}, \ldots, x_{r,m_k,\ldots,m_1})'$. Therefore, the unstructured covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}}$ (arbitrary but the same for all $r$) has $q = p_{1,k}\left(p_{1,k}+1\right)/2$ unknown parameters, which can be large for arbitrary values of either of the $m_j$'s. So, if the data is high-dimensional, i.e., if the sample size $n \leq q$, one cannot estimate the unknown covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}}$. Even if $n > q$ but relatively small the estimated covariance matrix becomes unstable. To circumvent this high-dimensionality or nearly high-dimensionality issues, it is then necessary to assume some appropriate structured variance-covariance matrix to reduce the number of unknown parameters, and $k-$SSCS covariance matrix (defined bellow in Definition 3) is a good choice for many high-dimensional higher-order data when the exchangeable feature is present in the data. The number of unknown parameters to be estimated in such "$k-$SSCS covariance matrix" is $\frac{k}{2} m_1 \left(m_1+1\right)$, which is much less than $q$. We will now define the $k$ SSCS covariance matrix.

**Definition 3.** *We say that $\boldsymbol{x}_r$ has a $k-$SSCS covariance matrix if $\boldsymbol{\Gamma}_{\boldsymbol{x}_r} = \mathrm{Cov}\left[\boldsymbol{x}_r\right]$ is of the form*

$$\boldsymbol{\Gamma}_{\boldsymbol{x}} = \boldsymbol{\Gamma}_k = \left[\sum_{j=1}^{k-1} \boldsymbol{I}_{p_{j+1,k}} \otimes \boldsymbol{J}_{p_{2,j}} \otimes \left(\boldsymbol{U}_{k,j} - \boldsymbol{U}_{k,j+1}\right)\right] + \boldsymbol{J}_{p_{2,k}} \otimes \boldsymbol{U}_{k,k}, \tag{12}$$

*where $\boldsymbol{U}_{k,j}$, for $j = 1, \ldots, k$, are $m_1 \times m_1-$matrices called SSCS-component matrices, with the assumption that $\boldsymbol{J}_{p_{2,1}}$ is equal to the real number 1.*

An alternative expression to (12) is

$$\boldsymbol{\Gamma}_{\boldsymbol{x}} = \boldsymbol{\Gamma}_k = \left[\sum_{j=1}^{k-1} \boldsymbol{I}_{p_{j+1,k}} \otimes \boldsymbol{J}_{p_{2,j}} \otimes \boldsymbol{T}_{k,j}\right] + \bigotimes_{h=1}^{k-1} \boldsymbol{J}_{m_{k+1-h}} \boldsymbol{T}_{k,k}, \tag{13}$$

with the convention that $\boldsymbol{J}_{p_{2,1}} = 1$, and

$$\boldsymbol{T}_{k,k} = \boldsymbol{U}_{k,k}, \tag{14}$$

$$\text{and} \quad \boldsymbol{T}_{k,j} = \boldsymbol{U}_{k,j} - \boldsymbol{U}_{k,j+1} \quad \text{for} \quad j = 1, \ldots, k-1 \tag{15}$$

$$\text{or equivalently} \quad \boldsymbol{U}_{k,i} = \sum_{h=i}^{k} \boldsymbol{T}_{k,h} \quad \text{for} \quad j = 1, \ldots, k.$$

In particular,

1. If $k=2$, and $m_1=1$, then $\boldsymbol{U}_{2,j}$, for $j=1,2$, are real numbers and the $m_2\times 1-$ random vector $\boldsymbol{x}_r$ has the covariance matrix

$$\begin{aligned}\boldsymbol{\Gamma}_{\boldsymbol{x}} &= \boldsymbol{\Gamma}_2 = \boldsymbol{I}_{m_2}\otimes(\boldsymbol{U}_{2,1}-\boldsymbol{U}_{2,2})+\boldsymbol{J}_{m_2}\otimes\boldsymbol{U}_{2,2}\\ &= (\boldsymbol{U}_{2,1}-\boldsymbol{U}_{2,2})\,\boldsymbol{I}_{m_2}+\boldsymbol{U}_{2,2}\boldsymbol{J}_{m_2},\end{aligned}$$

which is an $m_2\times m_2$ compound symmetry (CS) covariance matrix.

2. If $k=2$, and $m_1>1$, then the $m_1m_2\times 1-$ random vector $\boldsymbol{x}_r$ has the covariance matrix

$$\boldsymbol{\Gamma}_{\boldsymbol{x}}=\boldsymbol{\Gamma}_2=\boldsymbol{I}_{m_2}\otimes(\boldsymbol{U}_{2,1}-\boldsymbol{U}_{2,2})+\boldsymbol{J}_{m_2}\otimes\boldsymbol{U}_{2,2},$$

which is the equicorrelated covariance matrix (Leiva, 2007), block exchangeable covariance matrix (Roy et al., 2015) or BCS covariance matrix (Rao, 1954).

3. If $k=3$, the $m_1m_2m_3\times 1-$ random vector $\boldsymbol{x}_r$ has the covariance matrix

$$\boldsymbol{\Gamma}_{\boldsymbol{x}}=\boldsymbol{\Gamma}_3=\boldsymbol{I}_{m_3m_2}\otimes(\boldsymbol{U}_{3,1}-\boldsymbol{U}_{3,2})+\boldsymbol{I}_{m_3}\otimes\boldsymbol{J}_{m_2}\otimes(\boldsymbol{U}_{3,2}-\boldsymbol{U}_{3,3})+\boldsymbol{J}_{m_3m_2}\otimes\boldsymbol{U}_{3,3},$$

which is the jointly equicorrelated covariance matrix $\big($Leiva (2007)$\big)$, doubly exchangeable covariance matrix (Roy and Leiva, 2007; Leiva and Roy, 2012), or DBCS covariance matrix.

The covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}}$ given in (12) is called $k-$self similar compound symmetry covariance matrix because if we consider the $p_{1,k}-$dimensional vector $\boldsymbol{x}=(x_{1,\ldots,1},\ldots,x_{m_k,\ldots,m_1})'$ with a $k-$SSCS covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}}$ and for each fixed $g\epsilon\{2,\ldots,k-1\}$ we also consider the partition of $\boldsymbol{x}$ in $p_{1,g}-$ subvectors, then its corresponding covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}}$ is partitioned in $p_{1,g}\times p_{1,g}-$ submatrices, which is $(k+1-g)-$ SSCS matrix $\boldsymbol{\Gamma}^*_{k+1-g}$ as follows

$$\begin{aligned}\boldsymbol{\Gamma}_{\boldsymbol{x}} &= \underset{p_{1,k}\times p_{1,k}}{\boldsymbol{\Gamma}_{\boldsymbol{x}}}=\boldsymbol{\Gamma}_k=\boldsymbol{\Gamma}^*_{k+1-g} \qquad (16)\\ &= \left[\sum_{j=1}^{k+1-g-1}\boldsymbol{I}_{p_{j+1,k+1-g}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\boldsymbol{U}^*_{k+1-g,j}-\boldsymbol{U}^*_{k+1-g,j+1}\right)\right]+\boldsymbol{J}_{p_{2,k+1-g}}\otimes\boldsymbol{U}^*_{k+1-g,k+1-g},\end{aligned}$$

where $\boldsymbol{U}^*_{k+1-g,1}$ is the $g-$ SSCS matrix given by

$$\boldsymbol{U}^*_{k+1-g,1} = \left\{\sum_{f=1}^{g-1}\boldsymbol{I}_{p_{j+1,g}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes(\boldsymbol{U}_{k,f}-\boldsymbol{U}_{k,f+1})\right\}+\boldsymbol{J}_{p_{2,k}}\otimes\boldsymbol{U}_{k,g}, \qquad (17)$$

$$\text{and}\quad \boldsymbol{U}^*_{k+1-g,j} = \left(\bigotimes_{h=1}^{g-1}\boldsymbol{J}_{m_{g+1-h}}\right)\otimes\boldsymbol{U}_{k,g+j-1}\quad\text{for}\quad j\epsilon\{2,\ldots,k+1-g\}. \qquad (18)$$

**Example 3:** Case $k = 3$, $g = 2$ :

This matrix $\mathbf{\Gamma}_{\boldsymbol{x}} = \mathbf{\Gamma}_3$ is the jointly equicorrelated matrix given in Leiva and Roy (2012) and Roy and Leiva (2007). In this case the expression of the $k = 3-$SSCS covariance matrix $\mathbf{\Gamma}_{\boldsymbol{x}} = \mathbf{\Gamma}_3$ partitioned in $p_{1,g} \times p_{1,g}-$ submatrices for $g = 2$, is the $k + 1 - g = 2 - SSCC$ matrix $\mathbf{\Gamma}_2^*$ using formula (16) and is given by

$$\mathbf{\Gamma}_{\boldsymbol{x}} = \mathbf{\Gamma}_3 = \mathbf{\Gamma}_2^* = \boldsymbol{I}_{m_3} \otimes \left(\boldsymbol{U}_{2,1}^* - \boldsymbol{U}_{2,2}^*\right) + \boldsymbol{J}_{m_3} \otimes \boldsymbol{U}_{2,2}^*,$$

while, using (17), the $g = 2 - SSCC$ matrix is given by

$$\boldsymbol{U}_{k+1-g,1}^* \quad = \quad \boldsymbol{U}_{2,1}^* = \boldsymbol{I}_{m_2} \otimes (\boldsymbol{U}_{3,1} - \boldsymbol{U}_{3,2}) + \boldsymbol{J}_{m_2} \otimes \boldsymbol{U}_{3,2},$$

and, using (18) for $j = 2$, $\boldsymbol{U}_{2,2}^*$ is given by

$$\boldsymbol{U}_{k+1-g,j}^* = \boldsymbol{U}_{2,2}^* = \left(\bigotimes_{h=1}^{2-1} \boldsymbol{J}_{m_{2+1-h}}\right) \otimes \boldsymbol{U}_{k,g+j-1} = \boldsymbol{J}_{m_2} \otimes \boldsymbol{U}_{3,3}.$$

**Example 4:** Case $k = 4$, $g = 2$ :

The covariance $\mathbf{\Gamma}_{\boldsymbol{x}}$ is now a $k = 4-$SSCS matrix $\mathbf{\Gamma}_{\boldsymbol{x}} = \mathbf{\Gamma}_4$, but it is partitioned in $p_{1,2} \times p_{1,2}-$ submatrices. Therefore, $\mathbf{\Gamma}_{\boldsymbol{x}}$ has the form of a $k + 1 - g = 3-$SSCS matrix $\mathbf{\Gamma}_3^*$ whose expression is obtained using formula (16) as follows

$$\begin{aligned}\mathbf{\Gamma}_{\boldsymbol{x}} &= \mathbf{\Gamma}_4 = \mathbf{\Gamma}_3^* \\ &= \boldsymbol{I}_{m_4} \otimes \boldsymbol{I}_{m_3} \otimes \left(\boldsymbol{U}_{3,1}^* - \boldsymbol{U}_{3,2}^*\right) + \boldsymbol{I}_{m_4} \otimes \boldsymbol{J}_{m_3} \otimes \left(\boldsymbol{U}_{3,2}^* - \boldsymbol{U}_{3,3}^*\right) + \boldsymbol{J}_{m_4} \otimes \boldsymbol{J}_{m3} \otimes \boldsymbol{U}_{3,3}^*,\end{aligned}$$

where, using (17), the $g = 2 - SSCC$ matrix is given by

$$\boldsymbol{U}_{k+1-g,1}^* \quad = \quad \boldsymbol{U}_{3,1}^* = \boldsymbol{I}_{m_2} \otimes (\boldsymbol{U}_{4,1} - \boldsymbol{U}_{4,2}) + \boldsymbol{J}_{m_2} \otimes \boldsymbol{U}_{4,2},$$

and, using (18) for $j = 2, 3$, we obtain first $\boldsymbol{U}_{3,2}^*$

$$\boldsymbol{U}_{k+1-g,j}^* \quad = \quad \boldsymbol{U}_{3,2}^* = \left(\bigotimes_{h=1}^{2-1} \boldsymbol{J}_{m_{2+1-h}}\right) \otimes \boldsymbol{U}_{k,g+j-1} = \boldsymbol{J}_{m_2} \otimes \boldsymbol{U}_{4,3},$$

and then $\boldsymbol{U}_{3,3}^*$ as follows

$$\boldsymbol{U}_{k+1-g,j}^* \quad = \quad \boldsymbol{U}_{3,3}^* = \left(\bigotimes_{h=1}^{2-1} \boldsymbol{J}_{m_{2+1-h}}\right) \otimes \boldsymbol{U}_{k,g+j-1} = \boldsymbol{J}_{m_2} \otimes \boldsymbol{U}_{4,4}.$$

**Example 5:** Case $k = 4$, $g = 3$ :

The $k=4-$SSCS covariance $\mathbf{\Gamma}_{\boldsymbol{x}}$, when partitioned in $p_{1,3}\times p_{1,3}-$ submatrices (instead of $m_1\times m_1-$matrices) is a $k+1-g=2-SSCC$ matrix using formula (16), that is

$$\mathbf{\Gamma}_{\boldsymbol{x}}=\mathbf{\Gamma}_4=\mathbf{\Gamma}_2^*=\boldsymbol{I}_{m_4}\otimes\left(\boldsymbol{U}_{2,1}^*-\boldsymbol{U}_{2,2}^*\right)+\boldsymbol{J}_{m_4}\otimes\boldsymbol{U}_{2,2}^*,$$

where, using (17), the $g=2-SSCC$ matrix $\mathbf{\Gamma}_2^*$ is given by

$$\begin{aligned}&\boldsymbol{U}_{k+1-g,1}^*=\boldsymbol{U}_{2,1}^*\\ =\ &\boldsymbol{I}_{m_4}\otimes\boldsymbol{I}_{m_3}\otimes(\boldsymbol{U}_{4,1}-\boldsymbol{U}_{4,2})+\boldsymbol{I}_{m_4}\otimes\boldsymbol{J}_{m_3}\otimes(\boldsymbol{U}_{4,2}-\boldsymbol{U}_{4,3})+\boldsymbol{J}_{m_4}\otimes\boldsymbol{J}_{m3}\otimes\boldsymbol{U}_{4,3},\end{aligned}$$

and, using (18) for $j=2$, we obtain $\boldsymbol{U}_{2,2}^*$ as

$$\boldsymbol{U}_{k+1-g,j}^*\ =\ \boldsymbol{U}_{2,2}^*=\left(\bigotimes_{h=1}^{3-1}\boldsymbol{J}_{m_{3+1-h}}\right)\otimes\boldsymbol{U}_{k,g+j-1}=\boldsymbol{J}_{m_3}\otimes\boldsymbol{J}_{m_2}\otimes\boldsymbol{U}_{4,4}.$$

This self similar property of the matrix $\mathbf{\Gamma}_{\boldsymbol{x}}$ explained by (16), (17) and (18) in conjunction with the principle of Mathematical Induction can be used to prove the existence of $\mathbf{\Gamma}_{\boldsymbol{x}}^{-1}$ and to derive it's expression as well. For the expression of $\mathbf{\Gamma}_{\boldsymbol{x}}^{-1}$ we need matrices $\mathbf{\Delta}_{k,j}$, for $j=1,\ldots,k$, which are defined by

$$\mathbf{\Delta}_{k,j}=\sum_{i=1}^{j}p_{2,i}\left(\boldsymbol{U}_{k,i}-\boldsymbol{U}_{k,i+1}\right),\tag{19}$$

where $\boldsymbol{U}_{k,k+1}=\mathbf{0}$ and $p_{2,1}=1$. Note that

$$\mathbf{\Delta}_{k,j}=\begin{cases}\boldsymbol{U}_{k,1}-\boldsymbol{U}_{k,2} & if \quad j=1\\ \mathbf{\Delta}_{k,j-1}+p_{2,j}\left(\boldsymbol{U}_{k,j}-\boldsymbol{U}_{k,j+1}\right) & if \quad j=2,\ldots,k\end{cases},$$

where $\boldsymbol{U}_{k,k+1}=\mathbf{0}$**.** It can be proved that if matrices $\mathbf{\Delta}_{k,j}$, for $j=1,\ldots,k$ are non-singular, then

$$\mathbf{\Gamma}_{\boldsymbol{x}_r}^{-1}\ =\ \left[\sum_{j=1}^{k-1}\boldsymbol{I}_{p_{j+1,k}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\frac{1}{p_{2,j}}\left(\mathbf{\Delta}_{k,j}^{-1}-\mathbf{\Delta}_{k,j-1}^{-1}\right)\right]+\boldsymbol{J}_{p_{2,k}}\otimes\frac{1}{p_{2,k}}\left(\mathbf{\Delta}_{k,k}^{-1}-\mathbf{\Delta}_{k,k-1}^{-1}\right),$$

(see also Part 1 of Lemma 2 below) where the symbol $\mathbf{\Delta}_{k,0}^{-1}$ indicates the $m_1\times m_1$ zero matrix ($\mathbf{\Delta}_{k,0}^{-1}=\mathbf{0}_{m_1\times m_1}$). It is worthwhile to note that the structure of $\mathbf{\Gamma}_{\boldsymbol{x}_r}^{-1}$ is the same as the structure of $\mathbf{\Gamma}_{\boldsymbol{x}_r}$, that is, it has the $k-SSCS$ structure given in (13) with (14) and (15) and

$$\mathbf{\Gamma}_{\boldsymbol{x}_r}^{-1}=\mathbf{\Gamma}_k^{-1}=\left[\sum_{j=1}^{k-1}\boldsymbol{I}_{p_{j+1,k}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\boldsymbol{T}_{k,j}\right]+\boldsymbol{J}_{p_{2,k}}\otimes\boldsymbol{T}_{k,k},$$

where in this formula $\mathbf{\Gamma}_k^{-1}$, $\boldsymbol{T}_{k,j}$ is

$$\boldsymbol{T}_{k,j}=\frac{1}{p_{2,j}}\left(\mathbf{\Delta}_{k,j}^{-1}-\mathbf{\Delta}_{k,j-1}^{-1}\right),\quad \text{for each}\quad j=1,\ldots,k.$$

Using a similar inductive arguments, it can be proved that

$$|\mathbf{\Gamma}_{\boldsymbol{x}_r}| = |\mathbf{\Gamma}_k| = \prod_{j=1}^{k} |\mathbf{\Delta}_{k,j}|^{p_{j+1,k}-p_{j+2,k}},$$

where $\mathbf{\Delta}_{k,j}$ is given by (19) and it is assumed that $p_{k+1,k} = 1$ and $p_{k+2,k} = 0$. We will show in Lemma 4 that the matrices $\mathbf{\Delta}_{k,j}, j = 1, \ldots, k$ are the $k$ eigenblocks of the $k$ SSCS covariance structure. Moreover, we have the following Lemma.

**Lemma 2.** *Any $k - SSCS$ covariance matrix $\mathbf{\Gamma}_{\boldsymbol{x}} = \mathbf{\Gamma}_k$ satisfies the following properties:*

1. *$\mathbf{\Gamma}_k$ is a positive definite matrix if and only if the eigenblocks $\mathbf{\Delta}_{k,j}$, for $j = 1, \ldots, k$, are all positive definite matrices.*

2. *$\mathbf{\Gamma}_k$ can be written as*

$$\mathbf{\Gamma}_k = \boldsymbol{I}_{p_{2,k}} \otimes \boldsymbol{U}_{k,1} - \sum_{j=1}^{k-1} \boldsymbol{R}^{**}_{k,j+1} \otimes \boldsymbol{U}_{k,j+1}, \tag{20}$$

*where using notation (7), for $j = 1, \ldots, k-1$,*

$$\begin{aligned} \boldsymbol{R}^{**}_{k,j+1} &= \left( \bigotimes_{h=1}^{k-(j+1)} \boldsymbol{I}_{m_{k+1-h}} \right) \otimes (\boldsymbol{I}_{m_{j+1}} - m_{j+1}\boldsymbol{P}_{m_{j+1}}) \otimes \left( \bigotimes_{h=k-(j-1)}^{k-1} m_{k+1-h}\boldsymbol{P}_{m_{k+1-h}} \right) \\ &= \boldsymbol{I}_{p_{j+2,k}} \otimes (\boldsymbol{I}_{m_{j+1}} - m_{j+1}\boldsymbol{P}_{m_{j+1}}) \otimes p_{2,j}\boldsymbol{P}_{m_j,m_2} \\ &= \boldsymbol{I}_{p_{j+2,k}} \otimes (\boldsymbol{I}_{m_{j+1}} - \boldsymbol{J}_{m_{j+1}}) \otimes \boldsymbol{J}_{p_{j,2}}, \end{aligned}$$

*with $\bigotimes_{h=1}^{0} \boldsymbol{I}_{m_{k+1-h}} = 1 = \bigotimes_{h=k}^{k-1} m_{k+1-h}\boldsymbol{P}_{m_{k+1-h}}$.*

3. *The following equalities hold*

$$\boldsymbol{R}_{k,j+1}\mathbf{\Gamma}_k = \boldsymbol{R}^{*}_{k,j+1} \otimes \mathbf{\Delta}_{k,j} = \boldsymbol{R}_{k,j+1}\mathbf{\Gamma}_k\boldsymbol{R}_{k,j+1}, \tag{21}$$

*where $\boldsymbol{R}_{k,j+1}$ and $\boldsymbol{R}^{*}_{k,j+1}$ are given in (5) and (6) respectively, for each $j = 1, \ldots, k-1$, and*

$$\boldsymbol{R}_{k,k+1}\mathbf{\Gamma}_k = \boldsymbol{R}^{*}_{k,k+1} \otimes \mathbf{\Delta}_{k,k} = \boldsymbol{R}_{k,k+1}\mathbf{\Gamma}_k\boldsymbol{R}_{k,k+1}, \quad \textit{for} \quad j = k, \tag{22}$$

*where $\boldsymbol{R}_{k,k+1}$ and $\boldsymbol{R}^{*}_{k,k+1}$ are given in (8) and (9) respectively.*

4. *$\mathbf{\Gamma}_k$ can be written as the sum of $k$ orthogonal components as follows*

$$\mathbf{\Gamma}_k = \sum_{j=1}^{k} \boldsymbol{R}^{*}_{k,j+1} \otimes \mathbf{\Delta}_{k,j} \tag{23}$$

*and if* $\mathbf{\Gamma}_k^{-1}$ *exists, then it can be written as*

$$\mathbf{\Gamma}_k^{-1} = \sum_{j=1}^{k} \boldsymbol{R}_{k,j+1}^{*} \otimes \mathbf{\Delta}_{k,j}^{-1}, \tag{24}$$

*where* $\boldsymbol{R}_{k,j+1}^{*}$ *is given in (6), for each* $j = 1, \ldots, k-1$, *and* $\boldsymbol{R}_{k,k+1}^{*}$ *is given in (9) for* $j = k$.

*Proof.* 1. The proof of this property is posponed until we prove Lemma 4 in this present section.

2. It is clear that

$$\begin{aligned}\mathbf{\Gamma}_k &= \left[\sum_{j=1}^{k-1} \boldsymbol{I}_{p_{j+1,k}} \otimes \boldsymbol{J}_{p_{2,j}} \otimes (\boldsymbol{U}_{k,j} - \boldsymbol{U}_{k,j+1})\right] + \boldsymbol{J}_{p_{2,k}} \otimes \boldsymbol{U}_{k,k} \\ &= \boldsymbol{I}_{p_{2,k}} \otimes \boldsymbol{U}_{k,1} \\ &\quad - \left\{\sum_{j=1}^{k-1} \left[\left(\bigotimes_{h=1}^{k-(j+1)} \boldsymbol{I}_{m_{k+1-h}}\right) \otimes \left(\boldsymbol{I}_{m_{k+1-(k-j)}} - \boldsymbol{J}_{m_{k+1-(k-j)}}\right) \otimes \left(\bigotimes_{h=k-(j-1)}^{k-1} \boldsymbol{J}_{m_{k+1-h}}\right)\right] \boldsymbol{U}_{k,j+1}\right\}.\end{aligned}$$

3. Using properties of idempotency and mutual orthogonality of $\boldsymbol{P}_a$ and $\boldsymbol{Q}_a$ matrices in Definition 2, it is easy to prove that the $\boldsymbol{R}_{k,j+1}^{*} : j = 1, \ldots, k$ are symmetric idempotent matrices and orthogonal among them. Since the proof of the symmetry is trivial, we only prove

$$\boldsymbol{R}_{k,j+1}^{*} \boldsymbol{R}_{k,j^{*}+1}^{*} = \begin{cases} \boldsymbol{R}_{k,j+1}^{*} & if \quad j = j^{*} \\ \mathbf{0} & if \quad j \neq j^{*} \end{cases}. \tag{25}$$

Using notation (7) for $j > j^{*} = 1, \ldots, k$ (assuming that $\bigotimes_{h=1}^{0} \boldsymbol{I}_{m_{k+1-h}} = 1 = \bigotimes_{h=k+1}^{k-1} \boldsymbol{P}_{m_{k+1-h}} = \bigotimes_{h=k}^{k-1} P_{m_{k+1-h}}$) is

$$\begin{aligned}\boldsymbol{R}_{k,j^{*}+1}^{*} \boldsymbol{R}_{k,j+1}^{*} &= \left[\boldsymbol{I}_{p_{j+2,k}} \otimes \boldsymbol{Q}_{m_{j+1}} \otimes \boldsymbol{P}_{m_j, m_{j^{*}+2}} \otimes \boldsymbol{P}_{m_{j^{*}+1}} \otimes \boldsymbol{P}_{m_{j^{*}}, m_2}\right] \\ &\quad \left[\boldsymbol{I}_{p_{j+2,k}} \otimes \boldsymbol{I}_{m_{j+1}} \otimes \boldsymbol{I}_{p_{j^{*}+2,j}} \otimes \boldsymbol{Q}_{m_{j^{*}+1}} \otimes \boldsymbol{P}_{m_{j^{*}}, m_2}\right] \\ &= \mathbf{0},\end{aligned}$$

because $\boldsymbol{P}_{m_{j^{*}+1}} \boldsymbol{Q}_{m_{j^{*}+1}} = \boldsymbol{P}_{m_{j^{*}+1}} \left(\boldsymbol{I}_{m_{j^{*}+1}} - \boldsymbol{P}_{m_{j^{*}+1}}\right) = \mathbf{0}$, and $\boldsymbol{R}_{k,j+1}^{*} \boldsymbol{R}_{k,j+1}^{*} = \boldsymbol{R}_{k,j+1}^{*}$. Similarly, since

$$\begin{aligned}\boldsymbol{Q}_{m_{j+1}} \left(\boldsymbol{I}_{m_{j+1}} - m_{j+1} \boldsymbol{P}_{m_{j+1}}\right) &= \boldsymbol{I}_{m_{j+1}} - \boldsymbol{P}_{m_{j+1}} = \boldsymbol{Q}_{m_{j+1}}, \\ \boldsymbol{P}_{m_{j+1}} \left(\boldsymbol{I}_{m_{j+1}} - m_{j+1} \boldsymbol{P}_{m_{j+1}}\right) &= -\left(m_{j+1} - 1\right) \boldsymbol{P}_{m_{j+1}}, \quad \text{and} \\ \boldsymbol{Q}_{m_{j+1}} \left(m_{j+1} \boldsymbol{P}_{m_{j+1}}\right) &= \left(\boldsymbol{I}_{m_{j+1}} - \boldsymbol{P}_{m_{j+1}}\right) \left(m_{j+1} \boldsymbol{P}_{m_{j+1}}\right) = \mathbf{0},\end{aligned}$$

we have

$$\begin{aligned} \boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j+1} &= p_{2,j}\boldsymbol{R}^*_{k,j+1},\ \textit{for}\ j=1,\ldots,k-1, \\ \boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} &= -p_{2,j^*}\left(m_{j^*+1}-1\right)\boldsymbol{R}^*_{k,j+1},\ \textit{for}\ k\geq j>j^*=1,\ldots,k, \\ \boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} &= \boldsymbol{0},\ \textit{for}\ k-1\geq j^*>j=1,\ldots,k-2. \end{aligned}$$

It is trivial to prove that

$$\boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j+1} = p_{2,j}\boldsymbol{R}^*_{k,j+1}, \quad \text{for } j=1,\ldots,k-1, \tag{26}$$

so, let us first prove the following

$$\boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} = -p_{2,j^*}\left(m_{j^*+1}-1\right)\boldsymbol{R}^*_{k,j+1}, \quad \text{for } k\geq j>j^*=1,\ldots,k. \tag{27}$$

For $k\geq j>j^*=1,\ldots,k$, and assuming $\bigotimes_{h=1}^{0}\boldsymbol{I}_{m_{k+1-h}} = 1 = \bigotimes_{h=k+1}^{k-1}\boldsymbol{P}_{m_{k+1-h}} = \bigotimes_{h=k}^{k-1}\boldsymbol{P}_{m_{k+1-h}}$, is $\boldsymbol{I}_{p_{j+2,k}}$, we have

$$\begin{aligned} \boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} &= \left[\boldsymbol{I}_{p_{j+2,k}}\otimes\boldsymbol{Q}_{m_{j+1}}\otimes\boldsymbol{P}_{m_j,m_{j^*+2}}\otimes\boldsymbol{P}_{m_{j^*+1}}\otimes\boldsymbol{P}_{m_{j^*},m_2}\right] \\ &\quad \left[\boldsymbol{I}_{p_{j+2,k}}\otimes\boldsymbol{I}_{m_{j+1}}\otimes\boldsymbol{I}_{p_{j^*+2,j}}\otimes\left(\boldsymbol{I}_{m_{j^*+1}}-m_{j^*+1}\boldsymbol{P}_{m_{j^*+1}}\right)\otimes p_{2,j^*}\boldsymbol{P}_{m_{j^*},m_2}\right] \\ &= -\left(m_{j^*+1}-1\right)p_{2,j^*}\left[\boldsymbol{I}_{p_{j+2,k}}\otimes\boldsymbol{Q}_{m_{j+1}}\otimes\boldsymbol{P}_{m_j,m_{j^*+2}}\otimes\boldsymbol{P}_{m_{j^*+1}}\otimes\boldsymbol{P}_{m_{j^*},m_2}\right] \\ &= -p_{2,j^*}\left(m_{j^*+1}-1\right)\boldsymbol{R}^*_{k,j+1}. \end{aligned}$$

Finally, let's prove that

$$\boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} = \boldsymbol{0},\ \textit{for}\ k-1\geq j^*>j=1,\ldots,k-2. \tag{28}$$

For $k-1\geq j^*>j=1,\ldots,k-2$

$$\begin{aligned} &\boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} \\ &= \left[\boldsymbol{I}_{p_{j^*+2,k}}\otimes\boldsymbol{I}_{m_{j^*+1}}\otimes\boldsymbol{I}_{p_{j^*,j+2}}\otimes\boldsymbol{Q}_{m_{j+1}}\otimes\boldsymbol{P}_{m_j,m_2}\right] \\ &\quad \left[\boldsymbol{I}_{p_{j^*+2,k}}\otimes\left(\boldsymbol{I}_{m_{j^*+1}}-m_{j^*+1}\boldsymbol{P}_{m_{j^*+1}}\right)\otimes p_{j+2,j^*}\boldsymbol{P}_{m_{j^*},m_{j+2}}\otimes m_{j+1}\boldsymbol{P}_{m_{j+1}}\otimes p_{j,2}\boldsymbol{P}_{m_j,m_2}\right] \\ &= -\left[\boldsymbol{I}_{p_{j^*+2,k}}\otimes\left(\boldsymbol{I}_{m_{j^*+1}}-m_{j^*+1}\boldsymbol{P}_{m_{j^*+1}}\right)\otimes p_{j+2,j^*}\boldsymbol{P}_{m_{j^*},m_{j+2}}\otimes\boldsymbol{0}\otimes p_{j,2}\boldsymbol{P}_{m_j,m_2}\right] = \boldsymbol{0}. \end{aligned}$$

Therefore, to prove

$$\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k = \boldsymbol{R}^*_{k,j+1}\otimes\boldsymbol{\Delta}_{k,j},$$

let us start from the left hand side of the above equality. Multiplying the left side of (20) by $\boldsymbol{R}_{k,j+1} = \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{I}_{m_1}$, for $j = 1, \ldots, k$, we have

$$\begin{aligned}
\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k &= \left(\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{I}_{m_1}\right) \left[\boldsymbol{I}_{p_{2,k}} \otimes \boldsymbol{U}_{k,1} - \sum_{j^*=1}^{k-1} \boldsymbol{R}^{**}_{k,j^*+1} \otimes \boldsymbol{U}_{k,j^*+1}\right] \\
&= \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,1} - \left(\sum_{j^*=1}^{j-1} \boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} \otimes \boldsymbol{U}_{k,j^*+1}\right) \\
&\quad -\boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j+1} \otimes \boldsymbol{U}_{k,j+1} - \left(\sum_{j^*=j+1}^{k-1} \boldsymbol{R}^*_{k,j+1}\boldsymbol{R}^{**}_{k,j^*+1} \otimes \boldsymbol{U}_{k,j^*+1}\right),
\end{aligned}$$

then, using equalities (27), (26) and (28), it results

$$\begin{aligned}
\boldsymbol{R}_{k,j+1} &= \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,1} + \left(\sum_{j^*=1}^{j-1} p_{2,j^*}\left(m_{j^*+1} - 1\right)\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j^*+1}\right) \\
&\quad -p_{2,j}\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j+1} - \left(\sum_{j^*=j+1}^{k-1} \boldsymbol{0} \otimes \boldsymbol{U}_{k,j^*+1}\right) \\
&= \boldsymbol{R}^*_{k,j+1} \otimes \left(\boldsymbol{U}_{k,1} - \boldsymbol{U}_{k,2}\right) + \left(\sum_{j^{**}=2}^{j} p_{2,j^{**}}\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j^{**}}\right) \\
&\quad - \left(\sum_{j^*=2}^{j-1} p_{2,j^*}\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j^*+1}\right) - p_{2,j}\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j+1}.
\end{aligned}$$

Then, combining both the sums we get

$$\begin{aligned}
\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k &= \boldsymbol{R}^*_{k,j+1} \otimes \left(\boldsymbol{U}_{k,1} - \boldsymbol{U}_{k,2}\right) + \left[\sum_{j^*=2}^{j-1} p_{2,j^*}\boldsymbol{R}^*_{k,j+1} \otimes \left(\boldsymbol{U}_{k,j^*} - \boldsymbol{U}_{k,j^*+1}\right)\right] \\
&\quad +p_{2,j}\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j} - p_{2,j}\boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{U}_{k,j+1} \\
&= \sum_{j^*=1}^{j} p_{2,j^*}\boldsymbol{R}^*_{k,j+1} \otimes \left(\boldsymbol{U}_{k,j^*} - \boldsymbol{U}_{k,j^*+1}\right) \\
&= \boldsymbol{R}^*_{k,j+1} \otimes \sum_{j^*=1}^{j} p_{2,j^*}\left(\boldsymbol{U}_{k,j^*} - \boldsymbol{U}_{k,j^*+1}\right) = \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{\Delta}_{k,j}
\end{aligned}$$

Having proved

$$\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k = \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{\Delta}_{k,j}, \tag{29}$$

the set of second equalities

$$\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k\boldsymbol{R}_{k,j+1} = \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{\Delta}_{k,j},$$

is obtained by right multiplying the former equation (29) by $\boldsymbol{R}_{k,j+1} = \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{I}_{m_1}$, and using the idempotent property of the matrices $\boldsymbol{R}^*_{k,j+1}$ in (25).

4. It is easy to prove using Mathematical Induction that
$$\sum_{j=1}^{k} \boldsymbol{R}_{k,j+1} = \boldsymbol{I}_{p_{1,k}},$$
then using the previous Incise 3's result we obtain
$$\boldsymbol{\Gamma}_k = \left( \sum_{j=1}^{k} \boldsymbol{R}_{k,j+1} \right) \boldsymbol{\Gamma}_k = \sum_{j=1}^{k} \left( \boldsymbol{R}_{k,j+1} \boldsymbol{\Gamma}_k \right) = \sum_{j=1}^{k} \boldsymbol{R}^*_{k,j+1} \otimes \boldsymbol{\Delta}_{k,j} \, .$$
Finally, to prove a similar result for $\boldsymbol{\Gamma}_k^{-1}$ it is sufficient to multiply the right hand sides of (23) and (24) and show that the final product results in the identity matrix $\boldsymbol{I}_{p_{1,k}}$, using properties in (25).

□

Note that $\boldsymbol{\Gamma}_k$ given by (12) is a $k$ th order block matrix of the form (2), where

$$\underset{p_{1,h} \times p_{1,h}}{\boldsymbol{A}_{\boldsymbol{f}_{k,h+1};\boldsymbol{f}^*_{k,h+1}}} = \boldsymbol{A}_{f_k,f_{k-1},\ldots,f_{h+1};f^*_k,f^*_{k-1},\ldots,f^*_{h+1}}$$
$$= \begin{cases} \left[ \sum_{j=1}^{h-1} \boldsymbol{I}_{p_{j+1,h}} \otimes \boldsymbol{J}_{p_{2,j}} \otimes (\boldsymbol{U}_{k,j} - \boldsymbol{U}_{k,j+1}) \right] + \boldsymbol{J}_{p_{2,h}} \otimes (\boldsymbol{U}_{k,h} - \boldsymbol{U}_{k,h+1}) & if \quad \boldsymbol{f}_{k,h+1} = \boldsymbol{f}^*_{k,h+1} \\ \boldsymbol{J}_{p_{2,h}} \otimes (\boldsymbol{U}_{k,h} - \boldsymbol{U}_{k,h+1}) & if \quad \begin{matrix} \boldsymbol{f}_{k,h+2} = \boldsymbol{f}^*_{k,h+2} \\ f_{h+1} \neq f^*_{h+1} \end{matrix} \\ \vdots & \vdots \qquad \vdots \\ \boldsymbol{J}_{p_{2,h}} \otimes \boldsymbol{U}_{k,k} & if \quad f_k \neq f^*_k \end{cases},$$
that is, $\underset{p_{1,h} \times p_{1,h}}{\boldsymbol{A}_{\boldsymbol{f}_{k,h+1};\boldsymbol{f}_{k,h+1}}}$ is a $h - SSCS$ matrix with parameters $\boldsymbol{U}_{h,j} = \boldsymbol{U}_{k,j} - \boldsymbol{U}_{k,h+1}$, for $j = 1, \ldots, h$. Therefore, the results of Lemma 1 are valid for the $k - SSCS$ covariance matrix $\boldsymbol{\Gamma}_k$.

**Lemma 3.** *Let $\boldsymbol{\Gamma}_k$ be a $k - SSCS$ covariance matrix given by (12), then*

1. The following expressions of the parameters of $\boldsymbol{\Gamma}_k$ as a functions of $\boldsymbol{\Gamma}_k$ are useful.
$$\boldsymbol{U}_{k,1} = \frac{1}{q_{k,1}} BT_{p_{1,1}} \left( \boldsymbol{\Gamma}_k \right), \tag{30}$$
$$\boldsymbol{U}_{k,j} = \frac{1}{q_{k,j}} \left\{ BS_{p_{1,1}} \left[ BT_{p_{1,j}} \left( \boldsymbol{\Gamma}_k \right) \right] - BS_{p_{1,1}} \left[ BT_{p_{1,j-1}} \left( \boldsymbol{\Gamma}_k \right) \right] \right\}, \tag{31}$$
for $j = 2, \ldots, k$, where $BT_{p_{1,k}} \left( \boldsymbol{\Gamma}_k \right) = \boldsymbol{\Gamma}_k$ and
$$q_{k,j} = \begin{cases} \prod_{i=1}^{k-1} m_{k+1-i} = p_{2,k} & if \quad j = 1 \\ \left( \prod_{i=1}^{k-j} m_{k+1-i} \right) m_j (m_j - 1) \left( \prod_{i=k-(j-2)}^{k-1} m^2_{k+1-i} \right) & if \quad j = 2, \ldots, k \\ = p_{2,k} (m_j - 1) \, p_{2,j-1} & \end{cases}, \tag{32}$$

with $\prod_{i=k}^{k-1} m_{k+1-i} = 1$.

2. For $j = 2, \ldots, k$ it holds

$$\begin{aligned} BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k\right) &= BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k\boldsymbol{R}_{k,j+1}\right) \\ &= BT_{p_{1,1}}\left(\boldsymbol{R}^*_{k,j+1}\otimes\boldsymbol{\Delta}_{k,j}\right) = p_{j+2,k}\left(m_{j+1}-1\right)\boldsymbol{\Delta}_{k,j}, \\ \text{and}\quad BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1}\boldsymbol{\Gamma}_k\right) &= BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k\boldsymbol{R}_{k,j+1}\right) \\ &= BT_{p_{1,1}}\left(\boldsymbol{R}^*_{k,k+1}\otimes\boldsymbol{\Delta}_{k,j}\right) = \boldsymbol{\Delta}_{k,k}. \end{aligned}$$

*Proof.* 1. First note that $\boldsymbol{\Gamma}_k$ given in (10) is partitioned into $(p_{2,k})^2$ matrices, where $q_{k,g}$ of them are equal to $\boldsymbol{U}_{k,j}$, for each $g = 1, \ldots, k$. Using (14), that is, when $\boldsymbol{\Gamma}_k$ is partitioned into $p_{1,g}\times p_{1,g}$ matrices

$$\begin{aligned} \boldsymbol{\Gamma}_k &= \boldsymbol{\Gamma}^*_{k+1-g} = \left[\sum_{j=1}^{k+1-g-1}\boldsymbol{I}_{p_{j+1,k+1-g}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\boldsymbol{U}^*_{k+1-g,j}-\boldsymbol{U}^*_{k+1-g,j+1}\right)\right] \\ &\quad +\boldsymbol{J}_{p_{2,k+1-g}}\otimes\boldsymbol{U}^*_{k+1-g,k+1-g}, \end{aligned}$$

it can be seen that $BT_{p_{1,g}}(\boldsymbol{\Gamma}_k) = p_{g+1,k}\boldsymbol{U}^*_{k+1-g,1}$ where by (15) $\boldsymbol{U}^*_{k+1-g,1}$ is the $g-$ SSCS matrix given by

$$\boldsymbol{U}^*_{k+1-g,1} = \left\{\sum_{f=1}^{g-1}\boldsymbol{I}_{p_{j+1,g}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\boldsymbol{U}_{k,f}-\boldsymbol{U}_{k,f+1}\right)\right\}+\boldsymbol{J}_{p_{2,k}}\otimes\boldsymbol{U}_{k,g},$$

that includes all matrices $\boldsymbol{U}_{k,f} : f = 1, \ldots, g$ that are in $\boldsymbol{\Gamma}_k$, since all the others $\boldsymbol{U}_{k,f} : f = g+1, \ldots, k$ in $\boldsymbol{\Gamma}_k$ are, by (16), in matrices $\boldsymbol{U}^*_{k+1-g,j}$ outside the $p_{1,g}\times p_{1,g}-$ block diagonal of $\boldsymbol{\Gamma}_k$, that is, in

$$\boldsymbol{U}^*_{k+1-g,j} = \left(\bigotimes_{h=1}^{g-1}\boldsymbol{J}_{m_{g+1-h}}\right)\otimes\boldsymbol{U}_{k,g+j-1},$$

for $j = 2, \ldots, k+1-g$. It is clear then that only and all matrix $\boldsymbol{U}_{k,g}$ of $\boldsymbol{\Gamma}_k$ are added in $BS_{p_{1,1}}\left[BT_{p_{1,g}}(\boldsymbol{\Gamma}_k)\right] - BS_{p_{1,1}}\left[BT_{p_{1,g-1}}(\boldsymbol{\Gamma}_k)\right]$, therefore

$$BS_{p_{1,1}}\left[BT_{p_{1,g}}(\boldsymbol{\Gamma}_k)\right] - BS_{p_{1,1}}\left[BT_{p_{1,g-1}}(\boldsymbol{\Gamma}_k)\right] = q_{k,g}\boldsymbol{U}_{k,g}.$$

2. The first two equalities in this incise are trivial consequence of equations (21) and (22) of Part 3 of Lemma 2. For $j = 1, \ldots, k-1$,

$$\begin{aligned} BT_{p_{1,1}}\left(\boldsymbol{R}^*_{k,j+1}\otimes\boldsymbol{\Delta}_{k,j}\right) &= p_{j+1,k}p_{2,j}\frac{1}{p_{2,j}}\boldsymbol{\Delta}_{k,j} - p_{j+2,k}p_{2,j+1}\frac{1}{p_{2,j+1}}\boldsymbol{\Delta}_{k,j} \\ &= \left(p_{j+1,k}-p_{j+2,k}\right)\boldsymbol{\Delta}_{k,j} = p_{j+2,k}\left(m_{j+1}-1\right)\boldsymbol{\Delta}_{k,j}. \end{aligned}$$

Finally,

$$\begin{aligned} BT_{p_{1,1}}\left(\boldsymbol{R}^*_{k,k+1}\otimes\boldsymbol{\Delta}_{k,k}\right) &= BT_{p_{1,1}}\left[\left(\bigotimes_{h=1}^{k-1}\boldsymbol{P}_{m_{k+1-h}}\right)\otimes\boldsymbol{\Delta}_{k,k}\right] \\ &= p_{2,k}\frac{1}{p_{2,k}}\boldsymbol{\Delta}_{k,k}=\boldsymbol{\Delta}_{k,k}. \end{aligned}$$

□

In the next lemma we state that the $k-SSCS$ matrix $\boldsymbol{\Gamma}_k$ given by (12) can be transformed into a $m_1\times m_1-$block diagonal matrix (blocks in the diagonal are $m_1\times m_1-$matrices) by pre and post multiplying $\boldsymbol{\Gamma}_k$ by the appropriate matrices. It is known (see Leiva, 2007) that the $m_1\times m_1-$block diagonalization of the matrix $\boldsymbol{\Gamma}_2$

$$\boldsymbol{\Gamma}_2=\boldsymbol{I}_{m_2}\otimes(\underset{m_1\times m_1}{\boldsymbol{U}_{2,1}-\boldsymbol{U}_{2,2}})+\boldsymbol{J}_{m_2}\otimes\underset{m_1\times m_1}{\boldsymbol{U}_{2,2}}$$

is done by pre and post multiplying $\boldsymbol{\Gamma}_2$ by matrices as follows

$$\begin{aligned} \boldsymbol{L}_2'\boldsymbol{\Gamma}_2\boldsymbol{L}_2 &= \left(\boldsymbol{H}'_{m_2}\otimes\boldsymbol{I}_{m_1}\right)\boldsymbol{\Gamma}_2\left(\boldsymbol{H}_{m_2}\otimes\boldsymbol{I}_{m_1}\right) \\ &= \text{Diag}\left\{\boldsymbol{U}_{2,1}+(m_2-1)\,\boldsymbol{U}_{2,2};\boldsymbol{I}_{m_2-1}\otimes(\boldsymbol{U}_{2,1}-\boldsymbol{U}_{2,2})\right\}, \end{aligned}$$

where $\boldsymbol{H}_{m_2}$ is an $m_2\times m_2$ Helmert matrix. For the general $k$, the $k-$SSCS matrix given in (12), with the partition expression given in (16) for the case $g=k-1$, play an important role in the Mathematical Inductive proof implemented in Lemma 4. We particularize this formula here for $g=k-1$. This formula express $\boldsymbol{\Gamma}_k$ as a $2-SSCS$ matrix $\boldsymbol{\Gamma}_2^*$, that is,

$$\begin{aligned} \underset{p_{1,k}\times p_{1,k}}{\boldsymbol{\Gamma}_{\boldsymbol{x}}} &= \boldsymbol{\Gamma}_k=\boldsymbol{\Gamma}_2^* \\ &= \boldsymbol{I}_{m_k}\otimes\left(\boldsymbol{U}^*_{2,1}-\boldsymbol{U}^*_{2,2}\right)+\boldsymbol{J}_{m_k}\otimes\boldsymbol{U}^*_{2,2}, \end{aligned}$$

where $\boldsymbol{U}^*_{2,1}$ is the $(k-1)-$ SSCS matrix given by

$$\boldsymbol{U}^*_{2,1}=\left\{\sum_{j=1}^{k-2}\boldsymbol{I}_{p_{j+1,k-1}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes(\boldsymbol{U}_{k,j}-\boldsymbol{U}_{k,j+1})\right\}+\boldsymbol{J}_{p_{2,k-1}}\otimes\boldsymbol{U}_{k,k-1},$$

and $\boldsymbol{U}^*_{2,2}$ is given by

$$\boldsymbol{U}^*_{2,2}=\left(\bigotimes_{h=1}^{k-2}\boldsymbol{J}_{m_{k-h}}\right)\otimes\boldsymbol{U}_{k,k}=\boldsymbol{J}_{p_{2,k-1}}\otimes\boldsymbol{U}_{k,k}.$$

Also, the matrices $\boldsymbol{\Delta}^*_{2,1}=\boldsymbol{U}^*_{2,1}-\boldsymbol{U}^*_{2,2}$ and $\boldsymbol{\Delta}^*_{2,2}=\boldsymbol{U}^*_{2,1}+(m_k-1)\,\boldsymbol{U}^*_{2,2}=\boldsymbol{\Delta}^*_{2,1}+m_k\boldsymbol{U}^*_{2,2}$ will be useful in the proof of the inductive implication: *Case* $(k-1)\Rightarrow$ *Case* $k$, since both $\boldsymbol{\Delta}^*_{2,1}$ and $\boldsymbol{\Delta}^*_{2,2}$ are $(k-1)-SSCS$ matrices. Thus, we have the following result.

**Result 1.** *The matrix* $\underset{p_{1,k-1}\times p_{1,k-1}}{\boldsymbol{\Delta}^*_{2,1}}$ *is a* $(k-1)-$*SSCS matrix of the form*

$$\boldsymbol{U}^*_{2,1}-\boldsymbol{U}^*_{2,2}=\left[\sum_{j=1}^{k-2}\boldsymbol{I}_{p_{j+1,k-1}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\boldsymbol{U}^{(1)}_{k-1,j}-\boldsymbol{U}^{(1)}_{k-1,j+1}\right)\right]+\boldsymbol{J}_{p_{2,k-1}}\otimes\boldsymbol{U}^{(1)}_{k-,k-1},\tag{33}$$

*where*

$$\boldsymbol{U}^{(1)}_{k-1,j}=\boldsymbol{U}_{k,j}-\boldsymbol{U}_{k,k},\ \text{for } j=1,\ldots,k-1,\tag{34}$$

*and* $\underset{p_{1,k-1}\times p_{1,k-1}}{\boldsymbol{\Delta}^*_{2,2}}$ *is also a* $(k-1)-$*SSCS matrix of the form*

$$\boldsymbol{U}^*_{2,1}+(m_k-1)\,\boldsymbol{U}^*_{2,2}=\left[\sum_{j=1}^{k-2}\boldsymbol{I}_{p_{j+1,k-1}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\boldsymbol{U}^{(2)}_{k-1,j}-\boldsymbol{U}^{(2)}_{k-1,j+1}\right)\right]+\boldsymbol{J}_{p_{2,k-1}}\otimes\boldsymbol{U}^{(2)}_{k-,k-1},\tag{35}$$

*where*

$$\boldsymbol{U}^{(2)}_{k-1,j}=\boldsymbol{U}_{k,j}+(m_k-1)\,\boldsymbol{U}_{k,k},\quad \text{for } j=1,\ldots,k-1.\tag{36}$$

*Furthermore, for these two* $(k-1)-SSCS$ *matrices* $\boldsymbol{\Delta}^*_{2,1}$ *and* $\boldsymbol{\Delta}^*_{2,2}$ *it holds that their corresponding* $\boldsymbol{\Delta}'s$ *matrices are, respectively,*

$$\boldsymbol{\Delta}^{(1)}_{k-1,j}=\sum_{i=1}^{j}p_{2,i}\left(\boldsymbol{U}^{(1)}_{k,i}-\boldsymbol{U}^{(1)}_{k,i+1}\right)=\boldsymbol{\Delta}_{k,j},\quad \text{for}\quad j=1,\ldots,k-1,\tag{37}$$

$$\text{and}\quad \boldsymbol{\Delta}^{(2)}_{k-1,j}=\sum_{i=1}^{j}p_{2,i}\left(\boldsymbol{U}^{(2)}_{k,i}-\boldsymbol{U}^{(2)}_{k,i+1}\right)=\boldsymbol{\Delta}_{k,j},\quad \text{for}\quad j=1,\ldots,k-2,\tag{38}$$

$$\text{and}\quad \boldsymbol{\Delta}^{(2)}_{k-1,k-1}=\boldsymbol{\Delta}_{k,k},\tag{39}$$

*where* $p_{2,1}=1,$ *and* $\boldsymbol{\Delta}_{k,j}:j=1,\ldots,k$ *are the* $\boldsymbol{\Delta}'s$ *matrices of the original* $k-SSCS$ *matrix* $\boldsymbol{\Gamma}_k$*.*

*Proof.* We have

$$\begin{aligned}\boldsymbol{U}^*_{2,1}-\boldsymbol{U}^*_{2,2}=&\left\{\left[\sum_{j=1}^{k-2}\boldsymbol{I}_{p_{j+1,k-1}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes(\boldsymbol{U}_{k,j}-\boldsymbol{U}_{k,j+1})\right]+\boldsymbol{J}_{p_{2,k-1}}\otimes\boldsymbol{U}_{k,k-1}\right\}-\boldsymbol{J}_{p_{2,k-1}}\otimes\boldsymbol{U}_{k,k}\\=&\left[\sum_{j=1}^{k-2}\boldsymbol{I}_{p_{j+1,k-1}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes[(\boldsymbol{U}_{k,j}-\boldsymbol{U}_{k,k})-(\boldsymbol{U}_{k,j+1}-\boldsymbol{U}_{k,k})]\right]+\boldsymbol{J}_{p_{2,k-1}}\otimes(\boldsymbol{U}_{k,k-1}-\boldsymbol{U}_{k,k}),\end{aligned}$$

that is, by denoting $\boldsymbol{U}^{(1)}_{k-1,j}=\boldsymbol{U}_{k,j}-\boldsymbol{U}_{k,k}$, for $j=1,\ldots,k-1$ the required result is obtained. The proof of the second result is similar to the previous one.

Now, the $\boldsymbol{\Delta}'s$ matrices of the $(k-1)-SSCS$ matrix $\boldsymbol{\Delta}^*_{2,1} = \boldsymbol{U}^*_{2,1} - \boldsymbol{U}^*_{2,2}$, denoted by $\boldsymbol{\Delta}^{(1)}_{k-1,j}$ : $j = 1, \ldots, k-1$, are given by

$$\begin{aligned}\boldsymbol{\Delta}^{(1)}_{k-1,j} &= \sum_{i=1}^{j} p_{2,i}\left(\boldsymbol{U}^{(1)}_{k-1,i} - \boldsymbol{U}^{(1)}_{k-1,i+1}\right)\\ &= \sum_{i=1}^{j} p_{2,i}\left(\boldsymbol{U}_{k,i} - \boldsymbol{U}_{k,i+1}\right) = \boldsymbol{\Delta}_{k,j}, \quad \text{for} \quad j = 1, \ldots, k-2,\end{aligned}$$

$$\begin{aligned}\text{and} \quad \boldsymbol{\Delta}^{(1)}_{k-1,k-1} &= \left[\sum_{i=1}^{k-2} p_{2,i}\left(\boldsymbol{U}^{(1)}_{k-1,i} - \boldsymbol{U}^{(1)}_{k-1,i+1}\right)\right] + p_{2,k-1}\boldsymbol{U}^{(1)}_{k-1,k-1}\\ &= \sum_{i=1}^{k-1} p_{2,i}\left(\boldsymbol{U}_{k,i} - \boldsymbol{U}_{k,i+1}\right) = \boldsymbol{\Delta}_{k,k-1}.\end{aligned}$$

Similarly, the $\boldsymbol{\Delta}'s$ matrices of the $(k-1)-SSCS$ matrix $\boldsymbol{\Delta}^*_{2,2} = \boldsymbol{U}^*_{2,1} + (m_k - 1)\,\boldsymbol{U}^*_{2,2}$, denoted by $\boldsymbol{\Delta}^{(2)}_{k-1,j}$ : $j = 1, \ldots, k-1$, are given by

$$\begin{aligned}\boldsymbol{\Delta}^{(2)}_{k-1,j} &= \sum_{i=1}^{j} p_{2,i}\left(\boldsymbol{U}^{(2)}_{k-1,i} - \boldsymbol{U}^{(2)}_{k-1,i+1}\right)\\ &= \sum_{i=1}^{j} p_{2,i}\left(\boldsymbol{U}_{k,i} - \boldsymbol{U}_{k,i+1}\right) = \boldsymbol{\Delta}_{k,j}, \quad \text{for} \quad j = 1, \ldots, k-2,\end{aligned}$$

$$\begin{aligned}\text{and} \quad \boldsymbol{\Delta}^{(2)}_{k-1,k-1} &= \left[\sum_{i=1}^{k-2} p_{2,i}\left(\boldsymbol{U}^{(2)}_{k-1,i} - \boldsymbol{U}^{(2)}_{k-1,i+1}\right)\right] + p_{2,k-1}\boldsymbol{U}^{(2)}_{k-1,k-1}\\ &= \left[\sum_{i=1}^{k-1} p_{2,i}\left(\boldsymbol{U}_{k,i} - \boldsymbol{U}_{k,i+1}\right)\right] + p_{2,k}\boldsymbol{U}_{k,k} = \boldsymbol{\Delta}_{k,k}.\end{aligned}$$

□

Now we are ready to prove the $m_1 \times m_1-$block diagonalization of $\boldsymbol{\Gamma}_k$, where $k$ is any natural number with $k \geq 2$. For this we use the Mathematical Induction (M.I.) method, and we need to set some appropriate notations. For $1 \leq j < g < i \leq k$, let $\boldsymbol{I}_{m_i,m_j}$ denote the $m_i m_{i-1} \cdots m_j$ identity matrix, that is,

$$\boldsymbol{I}_{m_i,m_j} = \boldsymbol{I}_{m_i m_{i-1} \cdots m_j} = \boldsymbol{I}_{m_i} \otimes \boldsymbol{I}_{m_{i-1}} \otimes \cdots \otimes \boldsymbol{I}_{m_j},$$

and let

$$\underset{p_{g,i} \times p_{g,i}}{\boldsymbol{H}_{m_i,m_g}} = \boldsymbol{H}_{m_i} \otimes \boldsymbol{H}_{m_{i-1}} \otimes \cdots \otimes \boldsymbol{H}_{m_g},$$

where, for each $h = 2, \ldots, k$, $\boldsymbol{H}_{m_h}$ is an $m_h \times m_h$ Helmert matrix, that is, an orthogonal matrix whose first column is proportional to $\mathbf{1}_{m_h}$. Then

$$\boldsymbol{L}_h' = \boldsymbol{H}_{m_h,m_2}' \otimes \boldsymbol{I}_{m_1}$$

is an orthogonal matrix (note that $\boldsymbol{L}_h$ are not function of either of the $\boldsymbol{U}_i$'s), and in particular

$$\boldsymbol{L}_k' = \boldsymbol{H}_{m_h,m_2}' \otimes \boldsymbol{I}_{m_1} = \left(\boldsymbol{I}_{m_k} \otimes \boldsymbol{L}_{k-1}'\right)\left(\boldsymbol{H}_{m_k}' \otimes \boldsymbol{I}_{m_{k-1},m_1}\right). \tag{40}$$

Now we can have Lemma 4 that states the block diagonalization result of the SSCS matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}} = \boldsymbol{\Gamma}_k$.

**Lemma 4.** *The orthogonal matrix $\boldsymbol{L}_k'$ in (40) diagonalizes $\boldsymbol{\Gamma}_{\boldsymbol{x}}$, i.e.,*

$$\begin{aligned} \boldsymbol{L}_k'\boldsymbol{\Gamma}_{\boldsymbol{x}}\boldsymbol{L}_k &= \operatorname{Diag}\left\{\boldsymbol{D}_{\boldsymbol{f}}; \boldsymbol{f} = (f_k, f_{k-1}, \ldots, f_2)' \in F\right\} \\ &= \operatorname{Diag}\left\{\boldsymbol{D}_{f_k,f_{k-1},\ldots,f_2}; (f_k, f_{k-1}, \ldots, f_2)' \in F_k \times F_{k-1} \times \ldots \times F_2\right\}, \end{aligned} \tag{41}$$

*where for each $j = 1, \ldots, k$, the $m_1 \times m_1-$ diagonal matrices $\boldsymbol{D}_{\boldsymbol{f}} = \boldsymbol{D}_{f_k,f_{k-1},\ldots,f_2}$ are given by*

$$\boldsymbol{D}_{f_k,f_{k-1},\ldots,f_2} = \boldsymbol{\Delta}_{k,j} \quad if \quad f_2 = 1, \ldots, f_j = 1, f_{j+1} \neq 1,$$

*where $f_{k+1} \neq 1$ is not taken into consideration, that is,*

$$\boldsymbol{D}_{f_k,f_{k-1},\ldots,f_2} = \begin{cases} \boldsymbol{\Delta}_{k,k} & if \quad f_2 = 1, \ldots, f_{k-1} = 1, f_k = 1 \\ \boldsymbol{\Delta}_{k,k-1} & if \quad f_2 = 1, \ldots, f_{k-1} = 1, f_k \neq 1 \\ \boldsymbol{\Delta}_{k,k-2} & if \quad f_2 = 1, \ldots, f_{k-2} = 1, f_{k-1} \neq 1 \\ \vdots & \vdots \qquad \vdots \\ \boldsymbol{\Delta}_{k,2} & if \quad f_2 = 1, f_3 \neq 1 \\ \boldsymbol{\Delta}_{k,1} & if \quad f_2 \neq 1 \end{cases}. \tag{42}$$

*Thus, $\boldsymbol{\Delta}_{k,j}$, for $j = 1, \ldots, k$ are the $k$ eigenblocks of the $k$ SSCS covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{x}}$.*

*Proof.* We use M.I. to prove that (41), with $\boldsymbol{D}_{f_k,f_{k-1},\ldots,f_2}$ given in (42), is true for all natural numbers $k$. The proof of this result for $k = 2$ is given in Leiva (2007). So we assume that the result given in (41) is true for any $(k-1)-$SSCS matrix. In particular, we will use the inductive hypothesis for the $(k-1)-$SSCS matrices $\underset{p_{1,k-1}\times p_{1,k-1}}{\boldsymbol{\Delta}_{2,1}^*} = \boldsymbol{U}_{2,1}^* - \boldsymbol{U}_{2,2}^*$ and $\underset{p_{1,k-1}\times p_{1,k-1}}{\boldsymbol{\Delta}_{2,2}^*} = \boldsymbol{U}_{2,1}^* + (m_k - 1)\boldsymbol{U}_{2,2}^* = \boldsymbol{\Delta}_{2,1}^* + m_k\boldsymbol{U}_{2,2}^*$ defined in Result 1 by (33) and (35), respectively, and we will prove that (41) is true for an arbitrary $k-$SSCS matrix $\boldsymbol{\Gamma}_k$. Consider $\boldsymbol{\Gamma}_{\boldsymbol{x}} = \boldsymbol{\Gamma}_k$ an arbitrary $k-$SSCS matrix with SSCS-component matrices $\boldsymbol{U}_{k,j}$, $j = 1, \ldots, k$. Result 1 assures us that $\boldsymbol{\Gamma}_k$ can be written as

$$\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{\Gamma}_{\boldsymbol{x}}} = \boldsymbol{\Gamma}_k = \boldsymbol{\Gamma}_2^* = \boldsymbol{I}_{m_k} \otimes \left(\underset{p_{1,k-1}\times p_{1,k-1}}{\boldsymbol{U}_{2,1}^* - \boldsymbol{U}_{2,2}^*}\right) + \boldsymbol{J}_{m_k} \otimes \boldsymbol{U}_{2,2}^*,$$

where $\underset{p_{1,k-1}\times p_{1,k-1}}{\mathbf{\Delta}^*_{2,1}} = \boldsymbol{U}^*_{2,1} - \boldsymbol{U}^*_{2,2}$ and $\underset{p_{1,k-1}\times p_{1,k-1}}{\mathbf{\Delta}^*_{2,2}} = \boldsymbol{U}^*_{2,1} + (m_k - 1)\,\boldsymbol{U}^*_{2,2} = \mathbf{\Delta}^*_{2,1} + m_k\boldsymbol{U}^*_{2,2}$ are $(k-1)$ −SSCS matrices, given by (33) and (35) respectively, with $\boldsymbol{U}^{(1)}_{k-1,j}$ being defined in (34) and $\boldsymbol{U}^{(2)}_{k-1,j}$ in (36). Then, using the result for $k = 2$ given in Leiva (2007), we obtain

$$\begin{aligned} & \left(\boldsymbol{H}'_{m_k} \otimes \boldsymbol{I}_{m_{k-1},m_1}\right) \mathbf{\Gamma}_k \left(\boldsymbol{H}_{m_k} \otimes \boldsymbol{I}_{m_{k-1},m_1}\right) \\ = \;& \text{Diag}\left\{\boldsymbol{U}^*_{2,1} + (m_k - 1)\,\boldsymbol{U}^*_{2,2}; \boldsymbol{I}_{m_k-1} \otimes \left(\boldsymbol{U}^*_{2,1} - \boldsymbol{U}^*_{2,2}\right)\right\} \end{aligned} \tag{43}$$

By the inductive hypotesis, using $\boldsymbol{L}'_{k-1} = \boldsymbol{H}'_{m_{k-1}} \otimes \cdots \otimes \boldsymbol{H}'_{m_2} \otimes \boldsymbol{I}_{m_1} = \boldsymbol{H}'_{m_{k-1},m_2} \otimes \boldsymbol{I}_{m_1}$ we obtain

$$\begin{aligned} \boldsymbol{D}^{(1)}_{k-1} &= \boldsymbol{L}'_{k-1}\left(\boldsymbol{U}^*_{2,1} - \boldsymbol{U}^*_{2,2}\right)\boldsymbol{L}_{k-1} \\ &= \text{Diag}\left\{\boldsymbol{D}^{(1)}_{f_{k-1},\ldots,f_2}; (f_{k-1},\ldots,f_2)' \in F_{k-1} \times \ldots \times F_2\right\}, \end{aligned} \tag{44}$$

where using the expressions of $\mathbf{\Delta}^{(1)}_{k-1,j}$ for $j = 1, \ldots, k-1$ given in equations (37) in Result 1, the $m_1 \times m_1-$ blocks in the diagonal matrix $\boldsymbol{D}^{(1)}_{f_{k-1},\ldots,f_2}$ are as follows

$$\boldsymbol{D}^{(1)}_{f_{k-1},\ldots,f_2} = \begin{cases} \mathbf{\Delta}^{(1)}_{k-1,k-1} = \mathbf{\Delta}_{k,k-1} & if \quad f_2 = 1, \ldots, f_{k-1} = 1 \\ \mathbf{\Delta}^{(1)}_{k-1,k-2} = \mathbf{\Delta}_{k,k-2} & if \quad f_2 = 1, \ldots, f_{k-2} = 1, f_{k-1} \neq 1 \\ \mathbf{\Delta}^{(1)}_{k-1,k-3} = \mathbf{\Delta}_{k,k-3} & if \quad f_2 = 1, \ldots, f_{k-3} = 1, f_{k-2} \neq 1 \\ \quad\vdots & \;\vdots \qquad\qquad \vdots \\ \mathbf{\Delta}^{(1)}_{k-1,2} = \mathbf{\Delta}_{k,2} & if \quad f_2 = 1, f_3 \neq 1 \\ \mathbf{\Delta}^{(1)}_{k-1,1} = \mathbf{\Delta}_{k,1} & if \quad f_2 \neq 1 \end{cases}, \tag{45}$$

$$\text{and} \quad \begin{aligned} \boldsymbol{D}^{(2)}_{k-1} &= \boldsymbol{L}'_{k-1}\left(\boldsymbol{U}^*_{2,1} + (m_k - 1)\,\boldsymbol{U}^*_{2,2}\right)\boldsymbol{L}_{k-1} \\ &= \text{Diag}\left\{\boldsymbol{D}^{(2)}_{f_{k-1},\ldots,f_2}; (f_{k-1},\ldots,f_2)' \in F_{k-1} \times \ldots \times F_2\right\}, \end{aligned} \tag{46}$$

where, using the expressions of $\mathbf{\Delta}^{(2)}_{k-1,j}$ for $j = 1, \ldots, k-1$ given in equations (38) and (39) in Result 1, the $m_1 \times m_1-$ blocks in the diagonal matrix $\boldsymbol{D}^{(2)}_{f_{k-1},\ldots,f_2}$ are as follows

$$\boldsymbol{D}^{(2)}_{f_{k-1},\ldots,f_2} = \begin{cases} \mathbf{\Delta}^{(2)}_{k-1,k-1} = \mathbf{\Delta}_{k,k} & if \quad f_2 = 1, \ldots, f_{k-1} = 1 \\ \mathbf{\Delta}^{(2)}_{k-1,k-2} = \mathbf{\Delta}_{k,k-2} & if \quad f_2 = 1, \ldots, f_{k-2} = 1, f_{k-1} \neq 1 \\ \mathbf{\Delta}^{(2)}_{k-1,k-3} = \mathbf{\Delta}_{k,k-3} & if \quad f_2 = 1, \ldots, f_{k-3} = 1, f_{k-2} \neq 1 \\ \quad\vdots & \;\vdots \qquad\qquad \vdots \\ \mathbf{\Delta}^{(2)}_{k-1,2} = \mathbf{\Delta}_{k,2} & if \quad f_2 = 1, f_3 \neq 1 \\ \mathbf{\Delta}^{(2)}_{k-1,1} = \mathbf{\Delta}_{k,1} & if \quad f_2 \neq 1 \end{cases}. \tag{47}$$

Thus, pre multiplying by $(\boldsymbol{I}_{m_k} \otimes \boldsymbol{L}'_{k-1})$ and post multiplying by $(\boldsymbol{I}_{m_k} \otimes \boldsymbol{L}_{k-1})$ the matrix (43) we get

$$\begin{aligned} \boldsymbol{D}_k &= \left(\boldsymbol{I}_{m_k} \otimes \boldsymbol{L}_{k-1}'\right) \text{Diag}\left\{\boldsymbol{U}_{2,1}^* + (m_k - 1)\,\boldsymbol{U}_{2,2}^*; \boldsymbol{I}_{m_k - 1} \otimes \left(\boldsymbol{U}_{2,1}^* - \boldsymbol{U}_{2,2}^*\right)\right\} \left(\boldsymbol{I}_{m_k} \otimes \boldsymbol{L}_{k-1}\right) \\ &= \text{Diag}\left\{\boldsymbol{L}_{k-1}' \left[\boldsymbol{U}_{2,1}^* + (m_k - 1)\,\boldsymbol{U}_{2,2}^*\right] \boldsymbol{L}_{k-1}; \boldsymbol{I}_{m_k - 1} \otimes \boldsymbol{L}_{k-1}' \left(\boldsymbol{U}_{2,1}^* - \boldsymbol{U}_{2,2}^*\right) \boldsymbol{L}_{k-1}\right\} \\ &= \text{Diag}\left\{\boldsymbol{D}_{k-1}^{(2)}; \boldsymbol{I}_{m_k - 1} \otimes \boldsymbol{D}_{k-1}^{(1)}\right\}, \end{aligned}$$

where $\boldsymbol{D}_{k-1}^{(2)}$ and $\boldsymbol{D}_{k-1}^{(1)}$ are given in (46) with (47) and (44) with (45), respectively. That is, the $p_{1,k-1} \times p_{1,k-1}-$ matrix $\boldsymbol{D}_{k-1}^{(2)}$ appears in the first $p_{1,k-1}$ diagonal rows of $\boldsymbol{D}_k$ (i.e., when $f_k = 1$) and the $p_{1,k-1} \times p_{1,k-1}-$ matrix $\boldsymbol{D}_{k-1}^{(1)}$ appears in the last $(m_k - 1)\,p_{1,k-1}$ rows of $\boldsymbol{D}_k$ (i.e., when $f_k \neq 1$) and is repeated $(m_k - 1)$ times. □

**Note:** It is now clear the validity of Part 1 of Lemma 2. After the block diagonalization proved in Lemma 4, it is evident that $\boldsymbol{\Gamma}_k$ has all its eigenvalues positive if and only if all diagonal block matrices, i.e., the eigenblocks $\boldsymbol{\Delta}_{k,j}$, for each $j = 1, \ldots, k$, have also their its eigenvalues positive.

# 4 Estimators of the eigenblocks and their distributions

Let $\boldsymbol{x}_r : r = 1, \ldots, n$ be $p_{1,k} \times 1$ random vectors partitioned in $m_1 \times 1$ subvectors as

$$\begin{aligned} \boldsymbol{x}_r &= \left(\boldsymbol{x}_{r;\boldsymbol{f}}' : \boldsymbol{f} = (f_k, f_{k-1}, \ldots, f_2) \in F = F_{k,2} = \mathop{\times}_{j=k}^{2} F_j\right)' \\ &= \left(\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2}' : f_j \in F_j = \{1, \ldots, m_j\}\,,\ \text{for } j = k, k-1, \ldots, 2\right)'. \end{aligned}$$

These vectors are a random sample of size $n$ of a population with distribution $N_{p_{1,k}}\left(\boldsymbol{\mu}_{\boldsymbol{x}}; \boldsymbol{\Gamma}_{\boldsymbol{x}}\right)$, where $\boldsymbol{\Gamma}_{\boldsymbol{x}}$ is a positive definite $k-$SSCS structured covariance matrix as given in (12) in Definition 3. Let $\boldsymbol{X}$ be the $n \times p_{1,k}-$ sample data matrix as follows

$$\underset{n \times p_{1,k}}{\boldsymbol{X}} = \begin{pmatrix} \boldsymbol{x}_1' \\ \vdots \\ \boldsymbol{x}_n' \end{pmatrix} = \left(\underset{n \times m_1}{\boldsymbol{X}_{\cdot;1,\ldots,1}}, \ldots, \underset{n \times m_1}{\boldsymbol{X}_{\cdot;f_k,\ldots,f_2}}, \ldots, \underset{n \times m_1}{\boldsymbol{X}_{\cdot;m_k,\ldots,m_2}}\right)$$

with

$$\underset{n \times m_1}{\boldsymbol{X}_{\cdot,\boldsymbol{f}}} = \underset{n \times m_1}{\boldsymbol{X}_{\cdot,f_k,f_{k-1},\ldots,f_2}} = \begin{pmatrix} \boldsymbol{x}_{1,\boldsymbol{f}}' \\ \vdots \\ \boldsymbol{x}_{n,\boldsymbol{f}}' \end{pmatrix} = \left(\underset{1 \times m_1}{\boldsymbol{x}_{r,\boldsymbol{f}}'}\right)_{r=1}^{n}.$$

In this section we prove that certain unbiased estimators (to be defined) of the parameter matrices $\boldsymbol{U}_{k,j} : j = 1, \ldots, k-1$ can be written as functions of the usual sample covariance matrix $\boldsymbol{S}$ as follows

$$\begin{aligned}\boldsymbol{S} &= \frac{1}{n-1}\boldsymbol{X}'\boldsymbol{Q}_n\boldsymbol{X} \\ &= \frac{1}{n-1}\begin{pmatrix} \underset{m_1\times n}{\boldsymbol{X}'_{\cdot;1,\ldots,1}} \\ \vdots \\ \underset{m_1\times n}{\boldsymbol{X}'_{\cdot;m_k,\ldots,m_2}} \end{pmatrix}\boldsymbol{Q}_n\left(\underset{n\times m_1}{\boldsymbol{X}_{\cdot;1,\ldots,1}}, \ldots, \underset{n\times m_1}{\boldsymbol{X}_{\cdot;m_k,\ldots,m_2}}\right) = \left(\boldsymbol{S}_{\boldsymbol{f},\boldsymbol{f}^*}\right)_{\boldsymbol{f},\boldsymbol{f}^*\in F},\end{aligned}$$

where $\boldsymbol{Q}_n$ is given in (3) with (4). Now the sample mean $\overline{\boldsymbol{x}}$ can be expressed as

$$\begin{aligned}\underset{p_{1,k}\times 1}{\overline{\boldsymbol{x}}} &= \frac{1}{n}\underset{p_{1,k}\times n}{\boldsymbol{X}'}\boldsymbol{1}_n = \frac{1}{n}\begin{pmatrix} \boldsymbol{X}'_{\cdot;1,\ldots,1} \\ \vdots \\ \boldsymbol{X}'_{\cdot;m_k,\ldots,m_2} \end{pmatrix}\boldsymbol{1}_n = \begin{pmatrix} \frac{1}{n}\boldsymbol{X}'_{\cdot;1,\ldots,1}\boldsymbol{1}_n \\ \vdots \\ \frac{1}{n}\boldsymbol{X}'_{\cdot;m_k,\ldots,m_2}\boldsymbol{1}_n \end{pmatrix} \\ &= \left(\frac{1}{n}\underset{m_1\times n}{\boldsymbol{X}'_{\cdot;\boldsymbol{f}}}\boldsymbol{1}_n\right)_{\boldsymbol{f}\in F} = \left(\frac{1}{n}\sum_{r=1}^{n}\boldsymbol{x}_{r;\boldsymbol{f}}\right)_{\boldsymbol{f}\in F} = \underset{p_{1,k}\times 1}{\left(\underset{m_1\times 1}{\overline{\boldsymbol{x}}_{\boldsymbol{f}}}\right)_{\boldsymbol{f}\in F}}.\end{aligned}$$

Thus, $\boldsymbol{S}_{\boldsymbol{f},\boldsymbol{f}^*}$ in $\boldsymbol{S}$ can be expressed as

$$\boldsymbol{S}_{\boldsymbol{f},\boldsymbol{f}^*} = \frac{1}{n-1}\boldsymbol{X}'_{\cdot;\boldsymbol{f}}\boldsymbol{Q}_n\boldsymbol{X}'_{\cdot;\boldsymbol{f}^*} = \begin{cases} \frac{1}{n-1}\sum_{r=1}^{n}\left(\underset{m_1\times 1}{\boldsymbol{x}_{r;\boldsymbol{f}} - \overline{\boldsymbol{x}}_{\boldsymbol{f}}}\right)\left(\underset{1\times m_1}{\boldsymbol{x}'_{r;\boldsymbol{f}} - \overline{\boldsymbol{x}}'_{\boldsymbol{f}}}\right) & if \quad \boldsymbol{f} = \boldsymbol{f}^* \\ \frac{1}{n-1}\sum_{r=1}^{n}\left(\underset{m_1\times 1}{\boldsymbol{x}_{r;\boldsymbol{f}} - \overline{\boldsymbol{x}}_{\boldsymbol{f}}}\right)\left(\underset{1\times m_1}{\boldsymbol{x}'_{r;\boldsymbol{f}^*} - \overline{\boldsymbol{x}}'_{\boldsymbol{f}^*}}\right) & if \quad \boldsymbol{f} \neq \boldsymbol{f}^* \end{cases}.$$

We know that under the normality assumption

$$\boldsymbol{S}_n = \frac{n-1}{n}\,\boldsymbol{S} = \frac{1}{n}\boldsymbol{X}'\boldsymbol{Q}_n\boldsymbol{X}$$

is the maximum likelihood estimator of $\boldsymbol{\Gamma}_k$. Then we can easily find the MLEs of functions of $\boldsymbol{\Gamma}_k$ by using the MLE invariance property (see Mood et al., 1974 (Third edition, Theorem 2 on Page 285), Lehmann, E.L., 1983, Page 112; Anderson, T.W., 2003, Page 71, footnote). In particular, we use formulas (30) and (31) of Part 1 of the Lemma 3 to conclude that the following expressions, functions of the $k$ th order block matrix $\underset{p_{1,k}\times p_{1,k}}{\boldsymbol{S}_n}$, are the MLEs of the matrix parameters $\boldsymbol{U}_{k,j}$, for $j = 1, \ldots, k$ of $\boldsymbol{\Gamma}_k$ :

$$\begin{aligned}\widetilde{\boldsymbol{U}}_{k,1} &= \frac{blockTr_{p_{1,1},1}\left(\boldsymbol{S}_n\right)}{q_{k,1}},\ for\ j = 1 \\ \widetilde{\boldsymbol{U}}_{k,j} &= \frac{1}{q_{k,j}}\left\{BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{S}_n\right)\right] - BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{S}_n\right)\right]\right\} \quad \text{and for} \quad j = 2, \ldots, k,\end{aligned}$$

where $q_{k,j}$ is given by (32) in Lemma 3 and $\boldsymbol{S}_n$ is the MLE of $\boldsymbol{\Gamma}_{\boldsymbol{x}}$, that can be calculated using any statistical software package. Explicit expressions of the above results are given in the following lemma.

**Lemma 5.** *The MLE* $\widetilde{\boldsymbol{U}}_{k,k-j}$ *of* $\boldsymbol{U}_{k,k-j}$, *for* $j=0,\ldots,k-1$, *are expressed as*

$$\begin{aligned}
\widetilde{\boldsymbol{U}}_{k,1} &= \frac{blockTr_{p_{1,1},1}\left(\boldsymbol{S}_n\right)}{p_{2,k}} = \frac{1}{np_{2,k}}\sum_{r=1}^{n}\sum_{\boldsymbol{f}\in\boldsymbol{F}}\underset{m_1\times 1}{\left(\boldsymbol{x}_{r;\boldsymbol{f}}-\overline{\boldsymbol{x}}_{\boldsymbol{f}}\right)}\underset{1\times m_1}{\left(\boldsymbol{x}'_{r;\boldsymbol{f}}-\overline{\boldsymbol{x}}'_{\boldsymbol{f}}\right)} \\
&= \frac{1}{nq_{k,1}}\overbrace{\sum_{f_k\in\boldsymbol{F}_k}\cdots\sum_{f_2\in\boldsymbol{F}_2}}^{k-1\ sums}\sum_{r=1}^{n}\underset{m_1\times 1}{\left(\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2}-\overline{\boldsymbol{x}}_{f_k,f_{k-1},\ldots,f_2}\right)}\underset{m_1\times 1}{\left(\boldsymbol{x}'_{r;f_k,f_{k-1},\ldots,f_2}-\overline{\boldsymbol{x}}'_{f_k,f_{k-1},\ldots,f_2}\right)}
\end{aligned}$$

*and, for* $j=2,\ldots,k$,

$$\begin{aligned}
\widetilde{\boldsymbol{U}}_{k,j} &= \frac{1}{p_{2,k}\left(m_j-1\right)p_{2,j-1}}\left\{BS_{p_{1,1}}\left[BT_{p_{1,j}}\left(\boldsymbol{S}_n\right)\right]-BS_{p_{1,1}}\left[BT_{p_{1,j-1}}\left(\boldsymbol{S}_n\right)\right]\right\} \\
&= \frac{1}{nq_{k,k-j}}\overbrace{\sum_{f_k\in\boldsymbol{F}_k}\cdots\sum_{f_{j+1}\in\boldsymbol{F}_{j+1}}}^{k-j\ simple\ sums}\overbrace{\left(\sum_{f_j\in\boldsymbol{F}_j}\sum_{f_j\neq f_j^*\in\boldsymbol{F}_j}\right)}^{1\ special\ sum\ pair}\overbrace{\left(\sum_{f_{j-1}\in\boldsymbol{F}_{j-1}}\sum_{f_{j-1}^*\in\boldsymbol{F}_{j-1}}\right)\cdots\left(\sum_{f_2\in\boldsymbol{F}_k}\sum_{f_2^*\in\boldsymbol{F}_k}\right)}^{j-2\ sum\ pairs} \\
&\quad \sum_{r=1}^{n}\underset{m_1\times 1}{\left(\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2}-\overline{\boldsymbol{x}}_{f_k,f_{k-1},\ldots,f_2}\right)}\underset{1\times m_1}{\left(\boldsymbol{x}'_{r;f_k^*,f_{k-1}^*,\ldots,f_k^*}-\overline{\boldsymbol{x}}'_{f_k^*,f_{k-1}^*,\ldots,f_k^*}\right)}.
\end{aligned}$$

*Thus, the MLE of* $\boldsymbol{\Gamma}_{\boldsymbol{x}}=\boldsymbol{\Gamma}_k$, *is given by*

$$\widetilde{\boldsymbol{\Gamma}}_{\boldsymbol{x}}=\widetilde{\boldsymbol{\Gamma}}_k=\left[\sum_{j=1}^{k-1}\boldsymbol{I}_{p_{j+1,k}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\widetilde{\boldsymbol{U}}_{k,j}-\widetilde{\boldsymbol{U}}_{k,j+1}\right)\right]+\boldsymbol{J}_{p_{2,k}}\otimes\widetilde{\boldsymbol{U}}_{k,k}.$$

Since $\boldsymbol{S}=\frac{n}{n-1}\boldsymbol{S}_n$ is an unbiased estimator of $\boldsymbol{\Gamma}_{\boldsymbol{x}}$, we have

$$\begin{aligned}
E\left[\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}^*}\right] &= E\left[\boldsymbol{S}_{f_k,f_{k-1},\ldots,f_2;f_k^*,f_{k-1}^*,\ldots,f_2^*}\right] \\
&= \begin{cases}\boldsymbol{U}_{k,1} & if \qquad f_2=f_2^*,\ldots,f_k=f_k^* \\ \boldsymbol{U}_{k,j} & if \quad f_j\neq f_j^*,f_{j+1}=f_{j+1}^*,\ldots,f_k=f_k^* \quad for \quad j=2,\ldots,k\end{cases}.
\end{aligned}$$

Therefore, for each $j=1,\ldots,k$, it is natural to average all the above random matrices that are unbiased estimator of the same $\boldsymbol{U}_{k,j}$ to find a better unbiased estimator of $\boldsymbol{U}_{k,j}$. Thus, we have the following lemma.

**Lemma 6.** *Let* $\boldsymbol{\Gamma}_{\boldsymbol{x}}$ *be the* $k-$*SSCS variance-covariance matrix (as in equation (12) of Definition 3) with parameters* $\boldsymbol{U}_{k,j}:j=1,\ldots,k$ *and with the eigenblocks* $\boldsymbol{\Delta}_{k,j}:j=1,\ldots,k$ *given by (19). Then*

1. *For each* $j=1,\ldots,k$, $\widehat{\boldsymbol{U}}_{k,j}=\frac{n}{n-1}\widetilde{\boldsymbol{U}}_{k,j}$ *is an unbiased estimator of* $\boldsymbol{U}_{k,j}$. *Furthermore* $\widehat{\boldsymbol{U}}_{k,1}$ *can*

*be written as*

$$\begin{aligned}\widehat{\boldsymbol{U}}_{k,1} &= \frac{1}{p_{2,k}} BT_{p_{1,1}}(\boldsymbol{S}) = \frac{1}{p_{2,k}} \sum_{\boldsymbol{f}\in\boldsymbol{F}} \boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}} \\ &= \frac{1}{(n-1)\,q_{k,1}} \overbrace{\sum_{f_k\in F_k} \cdots \sum_{f_2\in F_2}}^{k-1\ sums} \sum_{r=1}^{n} \\ &\quad \left(\underset{m_1\times 1}{\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2} - \overline{\boldsymbol{x}}_{f_k,f_{k-1},\ldots,f_2}}\right)\left(\underset{m_1\times 1}{\boldsymbol{x}'_{r;f_k,f_{k-1},\ldots,f_2} - \overline{\boldsymbol{x}}'_{f_k,f_{k-1},\ldots,f_2}}\right)\end{aligned}$$

*and, for* $j=2,\ldots,k$, $\widehat{\boldsymbol{U}}_{k,j}$ *can be written as*

$$\begin{aligned}\widehat{\boldsymbol{U}}_{k,j} &= \frac{BS_{p_{1,1}}\left[BT_{p_{1,j}}(\boldsymbol{S})\right] - BS_{p_{1,1}}\left[BT_{p_{1,j-1}}(\boldsymbol{S})\right]}{q_{k,j}} \\ &= \frac{1}{(n-1)\,q_{k,j}} \overbrace{\sum_{f_k\in F_k} \cdots \sum_{f_{j+1}\in F_{j+1}}}^{k-j\ sums} \overbrace{\left(\sum_{f_j\in F_j} \sum_{f_j\neq f_j^*\in F_j}\right)}^{1\ special\ sum\ pair} \overbrace{\left(\sum_{f_{j-1}\in F_{j-1}} \sum_{f_{j-1}^*\in F_{j-1}}\right) \cdots \left(\sum_{f_2\in F_k} \sum_{f_2^*\in F_k}\right)}^{j-2\ sum\ pairs} \\ &\quad \sum_{r=1}^{n} \left(\underset{m_1\times 1}{\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2} - \overline{\boldsymbol{x}}_{f_k,f_{k-1},\ldots,f_2}}\right)\left(\underset{1\times m_1}{\boldsymbol{x}'_{r;f_k^*,f_{k-1}^*,\ldots,f_2^*} - \overline{\boldsymbol{x}}'_{f_k^*,f_{k-1}^*,\ldots,f_2^*}}\right),\end{aligned}$$

*where* $\boldsymbol{S} = \frac{1}{n-1}\boldsymbol{X}'\boldsymbol{Q}_n\boldsymbol{X}$ *and where* $q_{k,j}$ *is given by (32). Therefore, an unbiased estimator of* $\boldsymbol{\Gamma}_{\boldsymbol{x}} = \boldsymbol{\Gamma}_k$ *is given by*

$$\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{x}} = \widehat{\boldsymbol{\Gamma}}_k = \left[\sum_{j=1}^{k-1} \boldsymbol{I}_{p_{j+1,k}} \otimes \boldsymbol{J}_{p_{2,j}} \otimes \left(\widehat{\boldsymbol{U}}_{k,j} - \widehat{\boldsymbol{U}}_{k,j+1}\right)\right] + \boldsymbol{J}_{p_{2,k}} \otimes \widehat{\boldsymbol{U}}_{k,k}.$$

2. *For each* $j=1,\ldots,k$, *an unbiased estimator of the eigenblock* $\boldsymbol{\Delta}_{k,j}$, $\widehat{\boldsymbol{\Delta}}_{k,j}$ *is given by*

$$\widehat{\boldsymbol{\Delta}}_{k,j} = \sum_{i=1}^{j} p_{2,i}\left(\widehat{\boldsymbol{U}}_{k,i} - \widehat{\boldsymbol{U}}_{k,i+1}\right), \tag{48}$$

*where* $\widehat{\boldsymbol{U}}_{k,k+1} = \boldsymbol{0}$ *and* $p_{2,1}=1$, *or equivalently*

$$\widehat{\boldsymbol{\Delta}}_{k,j} = \begin{cases} \widehat{\boldsymbol{U}}_{k,1} - \widehat{\boldsymbol{U}}_{k,2} & if \quad j=1 \\ \widehat{\boldsymbol{\Delta}}_{k,j-1} + p_{2,j}\left(\widehat{\boldsymbol{U}}_{k,j} - \widehat{\boldsymbol{U}}_{k,j+1}\right) & if \quad j=2,\ldots,k \end{cases}, \tag{49}$$

*where* $\widehat{\boldsymbol{U}}_{k,k+1} = \boldsymbol{0}$. *The above unbiased estimators have the following expressions as functions of* $\boldsymbol{S}$,

$$\begin{aligned}\widehat{\boldsymbol{\Delta}}_{k,j} &= \frac{m_{j+1}\cdot BS_{p_{1,1}}\left[BT_{p_{1,j}}(\boldsymbol{S})\right] - BS_{p_{1,1}}\left[BT_{p_{1,j+1}}(\boldsymbol{S})\right]}{p_{2,k}\,(m_{j+1}-1)} \\ &= \frac{BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{S}\ \boldsymbol{R}_{k,j+1}\right)}{p_{j+2,k}\,(m_{j+1}-1)},\end{aligned} \tag{50}$$

*where* $BS_{p_{1,1}}\left[BT_{p_{1,1}}(\boldsymbol{S})\right]=BT_{p_{1,1}}(\boldsymbol{S})$ *and* $BS_{p_{1,1}}\left[BT_{p_{1,j+1}}(\boldsymbol{S})\right]=BS_{p_{1,1}}(\boldsymbol{S})$, *and an unbiased estimator of* $\boldsymbol{\Delta}_{k,k}$, $\widehat{\boldsymbol{\Delta}}_{k,k}$ *is given by*

$$\widehat{\boldsymbol{\Delta}}_{k,k} = \frac{1}{p_{2,k}}BS_{p_{1,1}}(\boldsymbol{S}) = BT_{p_{1,1}}(\boldsymbol{R}_{k,k+1}\boldsymbol{S}\ \boldsymbol{R}_{k,k+1}). \tag{51}$$

*Also,* $\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{x}}=\widehat{\boldsymbol{\Gamma}}_k$ *can be written as the following sum with orthogonal sumands*

$$\widehat{\boldsymbol{\Gamma}}_k = \sum_{j=1}^{k}\boldsymbol{R}^*_{k,j+1}\otimes\widehat{\boldsymbol{\Delta}}_{k,j}$$

*and if* $\widehat{\boldsymbol{\Gamma}}_k^{-1}$ *exists, it can be written as*

$$\widehat{\boldsymbol{\Gamma}}_k^{-1} = \sum_{j=1}^{k}\boldsymbol{R}^*_{k,j+1}\otimes\widehat{\boldsymbol{\Delta}}_{k,j}^{-1}, \tag{52}$$

*where* $\boldsymbol{R}_{k,j+1}$ *is given in (6), for each* $j=1,\ldots,k-1$, *and, for* $j=k$, $\boldsymbol{R}_{k,k+1}$ *is given in (9).*

*Proof.* The proof is easy. In particular,

1. Using Lemma 5 and the Expression of $E\left[\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}^*}\right]$ in Section 4 we get

$$\begin{aligned} E\left[\widehat{\boldsymbol{U}}_{k,1}\right] &= E\left[\frac{n}{n-1}\widetilde{\boldsymbol{U}}_{k,1}\right] \\ &= E\left[\frac{1}{p_{2,k}}BT_{p_{1,1}}(\boldsymbol{S})\right] = E\left[\frac{1}{p_{2,k}}\sum_{\boldsymbol{f}\in F}\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}}\right] = \boldsymbol{U}_{k,1}, \end{aligned}$$

but then

$$\begin{aligned} \widehat{\boldsymbol{U}}_{k,1} &= \frac{1}{p_{2,k}}\sum_{\boldsymbol{f}\in F}\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}} \\ &= \frac{1}{(n-1)\,q_{k,1}}\overbrace{\sum_{f_k\in F_k}\cdots\sum_{f_2\in F_2}}^{k-1\ sums}\sum_{r=1}^{n} \\ &\quad \left(\underset{m_1\times 1}{\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2}-\overline{\boldsymbol{x}}_{f_k,f_{k-1},\ldots,f_2}}\right)\left(\underset{m_1\times 1}{\boldsymbol{x}'_{r;f_k,f_{k-1},\ldots,f_2}-\overline{\boldsymbol{x}}'_{f_k,f_{k-1},\ldots,f_2}}\right). \end{aligned}$$

Similarly,

$$
\begin{aligned}
& E\left[\widehat{\boldsymbol{U}}_{k,j}\right] \\
= \; & E\left[\frac{n}{n-1}\widetilde{\boldsymbol{U}}_{k,j}\right] \\
= \; & E\left[\frac{BS_{p_{1,1}}\left[\left(BT_{p_{1,k-j}}\left(\boldsymbol{S}\right)\right)-BT_{p_{1,k-(j+1)}}\left(\boldsymbol{S}\right)\right]}{p_{2,k}\left(m_{k-j}-1\right)p_{2,k-(j+1)}}\right] \\
= \; & E\left[\frac{1}{q_{k,j}}\overbrace{\sum_{f_k\in F_k}\cdots\sum_{f_{j+1}\in F_{j+1}}}^{k-j\ sums}\overbrace{\left(\sum_{f_j\in F_j}\sum_{f_j\neq f_j^*\in F_j}\right)}^{1\ special\ sum\ pair}\overbrace{\left(\sum_{f_{j-1}\in F_{j-1}}\sum_{f_{j-1}^*\in F_{j-1}}\right)\cdots\left(\sum_{f_2\in F_k}\sum_{f_2^*\in F_k}\right)}^{j-2\ sum\ pairs}\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}^*}\right] \\
= \; & \frac{1}{q_{k,j}}\overbrace{\sum_{f_k\in F_k}\cdots\sum_{f_{j+1}\in F_{j+1}}}^{k-j\ sums}\overbrace{\left(\sum_{f_j\in F_j}\sum_{f_j\neq f_j^*\in F_j}\right)}^{1\ special\ sum\ pair}\overbrace{\left(\sum_{f_{j-1}\in F_{j-1}}\sum_{f_{j-1}^*\in F_{j-1}}\right)\cdots\left(\sum_{f_2\in F_k}\sum_{f_2^*\in F_k}\right)}^{j-2\ sum\ pairs}E\left[\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}^*}\right] \\
= \; & \boldsymbol{U}_{k,j},
\end{aligned}
$$

but then

$$
\begin{aligned}
& \widehat{\boldsymbol{U}}_{k,j} \\
= \; & \frac{1}{q_{k,k-j}}\overbrace{\sum_{f_k\in F_k}\cdots\sum_{f_{j+1}\in F_{j+1}}}^{k-j\ sums}\overbrace{\left(\sum_{f_j\in F_j}\sum_{f_j\neq f_j^*\in F_j}\right)}^{1\ special\ sum\ pair}\overbrace{\left(\sum_{f_{j-1}\in F_{j-1}}\sum_{f_{j-1}^*\in F_{j-1}}\right)\cdots\left(\sum_{f_2\in F_k}\sum_{f_2^*\in F_k}\right)}^{j-2\ sum\ pairs}\boldsymbol{S}_{\boldsymbol{f};\boldsymbol{f}^*} \\
= \; & \frac{1}{(n-1)\,q_{k,k-j}}\overbrace{\sum_{f_k\in\boldsymbol{F}_k}\cdots\sum_{f_{k-(j-1)}\in\boldsymbol{F}_{k-(j-1)}}}^{k-j\ simple\ sums}\overbrace{\left(\sum_{f_{k-j}\in\boldsymbol{F}_{k-j}}\sum_{f_{k-j}\neq f_{k-j}^*\in\boldsymbol{F}_{k-j}}\right)}^{1\ special\ sum\ pair} \\
& \overbrace{\left(\sum_{f_{k-(j+1)}\in\boldsymbol{F}_{k-(j+1)}}\sum_{f_{k-(j+1)}^*\in\boldsymbol{F}_{k-(j+1)}}\right)\cdots\left(\sum_{f_2\in\boldsymbol{F}_k}\sum_{f_2^*\in\boldsymbol{F}_k}\right)}^{j-2\ sum\ pairs} \\
& \sum_{r=1}^{n}\left(\underset{m_1\times 1}{\boldsymbol{x}_{r;f_k,f_{k-1},\ldots,f_2}-\overline{\boldsymbol{x}}_{f_k,f_{k-1},\ldots,f_2}}\right)\left(\underset{1\times m_1}{\boldsymbol{x}'_{r;f_k^*,f_{k-1}^*,\ldots,f_k^*}-\overline{\boldsymbol{x}}'_{f_k^*,f_{k-1}^*,\ldots,f_k^*}}\right).
\end{aligned}
$$

It is clear then that

$$
\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{x}}=\widehat{\boldsymbol{\Gamma}}_k=\left[\sum_{j=1}^{k-1}\boldsymbol{I}_{p_{j+1,k}}\otimes\boldsymbol{J}_{p_{2,j}}\otimes\left(\widehat{\boldsymbol{U}}_{k,j}-\widehat{\boldsymbol{U}}_{k,j+1}\right)\right]+\boldsymbol{J}_{p_{2,k}}\otimes\widehat{\boldsymbol{U}}_{k,k},
$$

is an unbiased estimator of $\boldsymbol{\Gamma}_{\boldsymbol{x}}=\boldsymbol{\Gamma}_k$.

2. From these results and (17) it follows that, for each $j = 1, \ldots, k$, an unbiased estimator of $\boldsymbol{\Delta}_{k,j}$ (given in (17) is $\widehat{\boldsymbol{\Delta}}_{k,j}$ given in (48) or (49). The first equality in (50) and (51) can be proved easily. Note that for $j = 1$,

$$\begin{aligned}\widehat{\boldsymbol{\Delta}}_{k,1} &= \widehat{\boldsymbol{U}}_{k,1} - \widehat{\boldsymbol{U}}_{k,2} \\ &= \frac{BT_{p_{11}}(\boldsymbol{S})}{p_{2,k}} - \frac{BS_{p_{1,1}}\left[BT_{p_{1,2}}(\boldsymbol{S})\right] - BS_{p_{1,1}}\left[BT_{p_{1,1}}(\boldsymbol{S})\right]}{p_{2,k}(m_2 - 1)} \\ &= \frac{m_2 BT_{p_{1,1}}(\boldsymbol{S}) - BS_{p_{1,1}}\left[BT_{p_{1,2}}(\boldsymbol{S})\right]}{p_{2,k}(m_2 - 1)},\end{aligned}$$

since $BS_{p_{1,1}}\left[BT_{p_{1,1}}(\boldsymbol{S})\right] = BT_{p_{1,1}}(\boldsymbol{S})$.

For $j = 2$,

$$\begin{aligned}\widehat{\boldsymbol{\Delta}}_{k,2} &= \widehat{\boldsymbol{\Delta}}_{k,1} + m_2\left(\widehat{\boldsymbol{U}}_{k,2} - \widehat{\boldsymbol{U}}_{k,3}\right) \\ &= \frac{(m_2 - 1)\, m_3}{p_{2,k}(m_2 - 1)(m_3 - 1)} BS_{p_{1,1}}\left[BT_{p_{1,2}}(\boldsymbol{S})\right] - \frac{1}{p_{2,k}(m_3 - 1)} BS_{p_{1,1}}\left[BT_{p_{1,3}}(\boldsymbol{S})\right] \\ &= \frac{1}{p_{2,k}(m_3 - 1)}\left\{m_3 BS_{p_{1,1}}\left[BT_{p_{1,2}}(\boldsymbol{S})\right] - BS_{p_{1,1}}\left[BT_{p_{1,3}}(\boldsymbol{S})\right]\right\}.\end{aligned}$$

It is clear that once it is proved for $k = j - 1$, that is,

$$\widehat{\boldsymbol{\Delta}}_{k,j-1} = \frac{m_j \cdot BS_{p_{1,1}}\left[BT_{p_{1,j-1}}(\boldsymbol{S})\right] - BS_{p_{1,1}}\left[BT_{p_{1,j}}(\boldsymbol{S})\right]}{p_{2,k}(m_j - 1)},$$

then using the relation $\widehat{\boldsymbol{\Delta}}_{k,j} = \widehat{\boldsymbol{\Delta}}_{k,j-1} + p_{2,j}\left(\widehat{\boldsymbol{U}}_{k,j} - \widehat{\boldsymbol{U}}_{k,j+1}\right)$ valid for $j = 2, \ldots, k-1$ and working in the same way as in the previous case, the validity of (50) can be proved. Analogously, using the expression already proved of $\widehat{\boldsymbol{\Delta}}_{k,k-1}$ and the relation $\widehat{\boldsymbol{\Delta}}_{k,k} = \widehat{\boldsymbol{\Delta}}_{k,k-1} + p_{2,k}\widehat{\boldsymbol{U}}_{k,k}$, the validity of (51) can be proved.

The proof of the other equalities included in (50), that is

$$\widehat{\boldsymbol{\Delta}}_{k,j} = \frac{BT_{p_{1,1}}(\boldsymbol{R}_{k,j+1}\boldsymbol{S})}{p_{j+2,k}(m_{j+1} - 1)} = \frac{BT_{p_{1,1}}(\boldsymbol{R}_{k,j+1}\boldsymbol{S}\ \boldsymbol{R}_{k,j+1})}{p_{j+2,k}(m_{j+1} - 1)}, \quad \text{for} \quad j = 1, \ldots, k,$$

where it is assumed that $p_{k+1,k}(m_{k+1} - 1) = 1$. This follows from (48) and Part 2 of Lemma 3.

□

Furthermore, we see that computation of both MLEs and the unbiased estimates of the component matrices is straightforward, as all of them have closed form solutions. Now, we proceed to the main result of this section that a multiple of the unbiased estimators of the eigenblocks $\widehat{\boldsymbol{\Delta}}_{k,j}$ have Wishart distributions for each $j = 1, \ldots, k$.

**Theorem 1.** *The random matrices*

$$(n-1)\,p_{j+2,k}\,(m_{j+1}-1)\,\widehat{\boldsymbol{\Delta}}_{k,j} = (n-1)\,BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{S}\ \boldsymbol{R}_{k,j+1}\right) : j = 1,\ldots,k-1,$$

$$\textit{and}\quad (n-1)\,\widehat{\boldsymbol{\Delta}}_{k,k} = (n-1)\,BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1}\boldsymbol{S}\ \boldsymbol{R}_{k,k+1}\right),$$

*where* $\boldsymbol{R}_{k,j+1} = \boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{I}_{m_1}$ *given by (5) with (6) and (8) with (9), are independent and*

$$\begin{aligned}(n-1)\,p_{j+2,k}\,(m_{j+1}-1)\,\widehat{\boldsymbol{\Delta}}_{k,j} &\sim W_{m_1}\left((n-1)\,p_{j+2,k}\,(m_{j+1}-1)\,;\boldsymbol{\Delta}_{k,j}\right)\quad \textit{for } 1,\ldots,k-1\\ \textit{and}\quad (n-1)\,\widehat{\boldsymbol{\Delta}}_{k,k} &\sim W_{m_1}\left((n-1)\,;\boldsymbol{\Delta}_{k,k}\right),\quad \textit{where } p_{k+1,k}=1.\end{aligned}$$

*Proof.* For each $\boldsymbol{f} = (f_k,\ldots,f_2) \in \mathop{\times}\limits_{h=1}^{k-1} F_{k+1-h} = F$, let $\boldsymbol{e}_{\boldsymbol{f};p_{2,k}}$ be the $p_{2,k}\times 1-$ vector that has a 1 in its $\boldsymbol{f}$ component and 0 otherwise. Then, for each $m_1\times 1-$vector $\boldsymbol{t}\neq\boldsymbol{0})$, and for each $j=1,\ldots,k$,

$$\begin{aligned}&(n-1)\,\boldsymbol{t}'BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{S}\ \boldsymbol{R}_{k,j+1}\right)\boldsymbol{t}\\ =\ &(n-1)\,\boldsymbol{t}'\left\{\sum_{f_k\in F_k}\cdots\sum_{f_2\in F_2}\left[\left(\boldsymbol{e}'_{\boldsymbol{f};p_{2,k}}\boldsymbol{R}^{*}_{k,j+1}\right)\otimes\boldsymbol{I}_{m_1}\right]\ \boldsymbol{S}\ \left[\left(\boldsymbol{e}_{\boldsymbol{f};p_{2,k}}\boldsymbol{R}^{*}_{k,j+1}\right)\otimes\boldsymbol{I}_{m_1}\right]\right\}\boldsymbol{t}\\ =\ &(n-1)\cdot\mathrm{tr}\left[\left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}'\right)\ \boldsymbol{S}\ \left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}\right)\right].\end{aligned}$$

Since by Equations (21) and (22) of Lemma 2, $\boldsymbol{R}_{k,j+1}\boldsymbol{\Gamma}_k\ \boldsymbol{R}_{k,j+1} = \boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{\Delta}_{k,j}$, we have

$$\begin{aligned}\left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}'\right)\ \boldsymbol{\Gamma}_k\ \left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}\right) &= \left(\boldsymbol{I}_{p_{2,k}}\otimes\boldsymbol{t}'\right)\left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{\Delta}_{k,j}\right)\cdot\left(\boldsymbol{I}_{p_{2,k}}\otimes\boldsymbol{t}\right)\\ &= \boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}'\boldsymbol{\Delta}_{k,j}\boldsymbol{t},\end{aligned}$$

that is, $\left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}'\right)\ \boldsymbol{\Gamma}_k\ \left(\boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}\right) = \boldsymbol{R}^{*}_{k,j+1}\otimes\boldsymbol{t}'\boldsymbol{\Delta}_{k,j}\boldsymbol{t}$ is a multiple of the idempotent matrix $\boldsymbol{R}^{*}_{k,j+1}$, then its only positive eigenvalue is $\boldsymbol{t}'\boldsymbol{\Delta}_{k,j}\boldsymbol{t}$ with multiplicity

$$\begin{aligned}r\left(\boldsymbol{R}^{*}_{k,j+1}\right) &= \begin{cases} r\left(\boldsymbol{I}_{p_{j+2,k}}\right)\otimes r\left(\boldsymbol{Q}_{m_{j+1}}\right)\otimes r\left(\boldsymbol{P}_{m_j,m_2}\right) & if\quad j=1,\ldots,k-1\\ r\left(\boldsymbol{P}_{m_k,m_2}\right) & if\quad j=k\end{cases}\\ &= \begin{cases} p_{j+2,k}\,(m_{j+1}-1) & if\quad j=1,\ldots,k-1\\ 1 & if\quad j=k\end{cases},\end{aligned}$$

where $\boldsymbol{P}_{m_j,m_2}$ is given in (7) and where we assume that $r\left(\boldsymbol{P}_{m_1,m_2}\right) = 1 = r\left(\boldsymbol{I}_{p_{k+1,k}}\right)$. Therefore, using Lemma 2 in Klein and Žežula (2010), for $j=1,\ldots,k-1$, we get

$$(n-1)\,\boldsymbol{t}'BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{S}\ \boldsymbol{R}_{k,j+1}\right)\boldsymbol{t}\sim\left(\boldsymbol{t}'\boldsymbol{\Delta}_{k,j}\boldsymbol{t}\right)\ \boldsymbol{\chi}^2\left((n-1)\,p_{j+2,k}\,(m_{j+1}-1)\right),$$

and then, we conclude that

$$(n-1)\,BT_{p_{1,1}}\left(\boldsymbol{R}_{k,j+1}\boldsymbol{S}\ \boldsymbol{R}_{k,j+1}\right)\sim W_{m_1}\left((n-1)\,p_{j+2,k}\,(m_{j+1}-1)\,;\boldsymbol{\Delta}_{k,j}\right),$$

or, equivalently, using (50),

$$(n-1)\, p_{j+2,k}\, (m_{j+1}-1)\, \widehat{\boldsymbol{\Delta}}_{k,j} \sim W_{m_1}\left((n-1)\, p_{j+2,k}\, (m_{j+1}-1)\,; \boldsymbol{\Delta}_{k,j}\right),$$

for $j=1,\ldots,k-1$, and similarly, for the case $j=k$, we get

$$(n-1)\, \boldsymbol{t}' BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1} \boldsymbol{S}\ \boldsymbol{R}_{k,k+1}\right) \boldsymbol{t} \sim \left(\boldsymbol{t}' \boldsymbol{\Delta}_{k,j} \boldsymbol{t}\right)\ \boldsymbol{\chi}^2\left((n-1)\right),$$

and then, we conclude that

$$(n-1)\, BT_{p_{1,1}}\left(\boldsymbol{R}_{k,k+1} \boldsymbol{S}\ \boldsymbol{R}_{k,k+1}\right) \sim W_{m_1}\left((n-1)\,; \boldsymbol{\Delta}_{k,k}\right),$$

or, equivalently, using (51),

$$(n-1)\, \widehat{\boldsymbol{\Delta}}_{k,k} \sim W_{m_1}\big((n-1), \boldsymbol{\Delta}_{k,k}\big).$$

The independence of these distributions is a consequence of Craig's theorem 1943 (see Theorem 3.4.5, Mardia et al (1982)) because $\boldsymbol{R}_{k,j+1} \cdot \boldsymbol{R}_{k,i+1} = \mathbf{0}$, if $i \neq j, i, j = 1, \ldots, k-1$. □

We will now derive the distributions of the eigenblocks for some special cases ($k=2$ and $k=3$) in the following corollary.

**Corollary 1.** *The 2 SSCS covariance matrix for 2nd order data or multivariate repeated measures data has two eigenblocks, $\boldsymbol{\Delta}_{2,2}$, and $\boldsymbol{\Delta}_{2,1}$ with multiplicity $m_2-1$. So, from Theorem 1 we get*

$$\begin{aligned}(n-1)\, p_{1+2,k}\, (m_{1+1}-1)\, \widehat{\boldsymbol{\Delta}}_{k,1} &\sim W_{m_1}\big((n-1)p_{1+2,k}(m_{1+1}-1), \boldsymbol{\Delta}_{k,1}\big)\\ \text{or}\ \ (n-1)\, (m_2-1)\, \widehat{\boldsymbol{\Delta}}_{2,1} &\sim W_{m_1}\big((n-1)(m_2-1), \boldsymbol{\Delta}_{2,1}\big)\ \text{as}\ p_{3,2}=1\ \text{by (1)}, \quad (53)\\ \text{and}\ \ (n-1)\, \widehat{\boldsymbol{\Delta}}_{2,2} &\sim W_{m_1}\big((n-1), \boldsymbol{\Delta}_{2,2}\big). \quad (54)\end{aligned}$$

*Thus, we see that the distributions of these two eigenblocks in (53) and (54) are identical to the distributions of the eigenblocks obtained by Roy et al. (2015) for 2 SSCS or BCS covariance structure for multivariate repeated measures data. Therefore, we can say that our result is an extension of Roy et al.'s (2015) distributions of eigenblocks for k SSCS covariance structure for k th order data.*

*The 3 SSCS covariance matrix for 3rd order data has three eigenblocks, $\boldsymbol{\Delta}_{3,3}$, $\boldsymbol{\Delta}_{3,2}$ with multiplicity $m_3-1$, and $\boldsymbol{\Delta}_{3,1}$ with multiplicity $m_3(m_2-1)$. So, from Theorem 1 we get*

$$\begin{aligned}(n-1)\, p_{1+2,k}\, (m_{1+1}-1)\, \widehat{\boldsymbol{\Delta}}_{k,1} &\sim W_{m_1}\big((n-1)p_{1+2,k}(m_{1+1}-1), \boldsymbol{\Delta}_{k,1}\big)\\ \text{or}\ \ (n-1)\, m_3\, (m_2-1)\, \widehat{\boldsymbol{\Delta}}_{3,1} &\sim W_{m_1}\big((n-1)m_3(m_2-1), \boldsymbol{\Delta}_{3,1}\big)\ \text{as}\ p_{3,3}=m_3\ \text{by (1)}, \quad (55)\\ \text{and}\ \ (n-1)\, p_{2+2,k}\, (m_{2+1}-1)\, \widehat{\boldsymbol{\Delta}}_{k,2} &\sim W_{m_1}\big((n-1)p_{2+2,k}(m_3-1), \boldsymbol{\Delta}_{k,2}\big)\\ \text{or}\ \ (n-1)\, (m_3-1)\, \widehat{\boldsymbol{\Delta}}_{3,2} &\sim W_{m_1}\big((n-1)(m_3-1), \boldsymbol{\Delta}_{3,2}\big)\ \text{as}\ p_{4,3}=1\ \text{by (1)}, \quad (56)\\ \text{and}\ \ (n-1)\, \widehat{\boldsymbol{\Delta}}_{3,3} &\sim W_{m_1}\big((n-1), \boldsymbol{\Delta}_{3,3}\big). \quad (57)\end{aligned}$$

Thus, we see that the distributions of the three eigenblocks in (55), (56) and (57) are exactly same as the distributions of the eigenblocks first obtained by Klein et al. (2014) for 3 SSCS or DBCS covariance structure for multivariate repeated measures data. Therefore, we can say that our result in this article is an extension or generalization of Klein et al.'s (2014) distributions of eigenblocks for $k$ SSCS covariance matrix for $k$ th order data.

## 5 Conclusions

We derived the unbiased estimates and the distribution of the eigenblocks of $k$ SSCS covariance structure. Distributions of the eigenblocks of $k$ SSCS covariance structure are needed for testing of equality of mean for two populations with array-variate data with $k$ SSCS covariance structure. The classical problems of clustering and discriminant analysis (Anderson, 2003) are exclusively based on precise knowledge of covariance matrices, and the first step of discriminant analysis is to test the equality of means for two populations. So, our new method will have multiple applications in the analyses of array-variate datasets.

## References


[1] Arnold SF (1979) Linear models with exchangeably distributed errors. J Am Stat Assoc 74:194–199.

[2] Anderson, T.W., 2003. An introduction to multivariate statistical analysis, Third Ed. John Wiley & Sons.

[3] Klein, D., Žežula, I. (2010). Orthogonal decompositions in growth curve models. Acta et Commentationes Universitatis Tartuensis de Mathematica 14: 35-44.

[4] Klein, D., Roy, A., Leiva, R., Žežula, I. (2014). Multivariate data with block compound symmetry covariance structure. The Paper Presented at the LinStat2014, Linköping, Sweden, August 24-28, 2014.

[5] Kozioł A., Roy A., Zmylony, R., Leiva, R. and Fonseca, M. (2018). Free-coordinate Estimation for Doubly Multivariate Data, Linear Algebra and its Applications, 547, 217- 239.

[6] Lehmann, E.L. (1983). Theory of point estimation, John Wiley and sons, Inc., N.Y. ISSN 0271-6232.

[7] Leiva, R. (2007). Linear discrimination with equicorrelated training vectors. Journal of Multivariate Analysis 98: 384-409.

[8] Leiva R., Roy, A. (2011). Linear Discrimination for Multi-level Multivariate Data with Separable Means and Jointly Equicorrelated Covariance Structure. Journal of Statistical Planning and Inference, 141(5): 1910-1924.

[9] Leiva R., Roy, A. (2012). Linear discrimination for three-level multivariate data with separable additive mean vector and doubly exchangeable covariance structure. Computational Statistics and Data Analysis, 56(6): 1644-1661.

[10] Mardia, K.V., Kent, J.T., Bibby, J.M. (1979). Multivariate Analysis. New York: Academic Press Inc.

[11] Mood, A., Graybill, F. and Boes, D. (1974). Itroduction to the Theory of Statistics, Third edition, Mc Graw-Hill, Inc.

[12] Rao, C.R. (1953). Discriminant functions for genetic differentiation and selection. Sankhya, 12: 229-246.

[13] Roy A. and Leiva R. (2007). Discrimination with Jointly Equicorrelated Multi-level Multivariate Data, Advances in Data Analysis and Classification, 1(3), 175-199.

[14] Roy, A., Leiva, R. (2011). Estimating and Testing a Structured Covariance Matrix for Three-level Multivariate Data. Communications in Statistics - Theory and Methods, 40(10).

[15] Roy, A., and Fonseca, M. (2012), Linear Models with Doubly Exchangeable Distributed Errors. Communications in Statistics - Theory and Methods, 41(13), 2545-2569.

[16] Žežula I. and Klein D. and Roy, A. (2018) Testing of Multivariate Repeated Measures Data with Block Exchangeable Covariance Structure, TEST, 27(2), 360-378.